Joint Power and Subcarrier Allocation in Multi-Cell Multi-Carrier NOMA

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***Abstract*—Non-orthogonal multiple access (NOMA) is a technology proposed for next generation cellular networks because of its high spectral efficiency and enhanced user connectivity. However, in the literature the optimal joint power and sub-carrier allocation for NOMA has been proposed for single cell only. Consequently, a global optimal algorithm for the joint power and sub-carrier allocation for NOMA system in multi-cell scenario is still an open problem. In this work, we propose a polyblock optimization based algorithm for obtaining a global optimal solution. It has reduced complexity due to a necessary and sufficient condition for feasible successive interference cancellation (SIC). Besides, we can adjust its optimization approximation parameter to serve as benchmark solution or to provide suitable solution for multi-cell multi-carrier NOMA systems. Numerical studies have shown its effectiveness.**

I. Introduction

Orthogonal frequency division multiple access (OFDMA) is the downlink multiplexing scheme adopted by 5G New Radio. In each cell, every sub-carrier is allocated to at most one user such that intra-cell interference is almost suppressed. However, OFDMA is known to be sub-optimal in spectral efficiency [1]. Power domain non-orthogonal multiple access (NOMA) is a capacity-achieving multiple access scheme based on successive interference cancellation (SIC) that has been proposed for future mobile networks. In NOMA, unlike orthogonal multiple access (OMA) such as OFDMA, each sub-carrier can be allocated to possibly more than one user; multiple users with diverse power levels can be accommodated in the same resource block with the aid of superposition coding and SIC.

The intra-cell interference occurs since multiple users in the same cell can be allocated with the same sub-carrier. Note that the resource allocation optimization problem for single cell NOMA has been addressed extensively in the literature, see e.g., [2], [3] and the references therein. In this paper, we would focus on the optimization problem for multi-cell NOMA. We consider a frequency reuse-1 system, i.e., every cell shares the same sub-carriers when performing NOMA, resulting in inter-cell interference. The presence of inter-cell interference with intra-cell interference in such multi-cell NOMA system makes the resource allocation problem more difficult.

In [4], the authors addressed the uplink precoder design optimization problem for multi-cell MIMO-NOMA and performed sum rate maximization using an approximate algorithm. In [5], [6], the authors employed monotonic optimization based methods which aim to reach a global optimal solution. The paper [5] addressed the global optimal power control problem in wireless networks over multiple interfering links. In [6], the power and sub-carrier allocation problem for sum-rate maximization in multi-cell OFDMA systems has been investigated.

In [7], the authors consider a multi-cell single carrier CoMP-NOMA system and provide a power allocation solution. In [8], the power minimization problem for a downlink NOMA multi-cell system subject to user minimum data rate requirement is addressed. However, the sum rate maximization problem is not investigated. In [9], heuristic algorithms for the power allocation in multi-cell multi-carrier NOMA systems for sum power minimization as well as sum rate maximization have been devised.

To the best of our knowledge, so far no one has proposed a global optimal solution for the multi-cell multi-carrier NOMA join power and sub-carrier allocation problem for the sum rate maximization. Here, we aim to solve the above problem. We propose a polyblock optimization based algorithm, which has been used extensively in many resource allocation problems, see e.g., [5], [6]. We extend this method to solve the NOMA joint optimization problem in multi-cell setup. In the meantime, we derive the necessary and sufficient condition for feasible SIC in multi-cell NOMA, which can be used for the class of problems to reduce the optimization complexity.

II. System Model and Problem Formulation

We consider a multi-cell multi-carrier NOMA downlink system, which consists of K base stations (BS) denoted by the set $\mathcal{K}$. The set of users served by a BS $k \in \mathcal{K}$ is denoted by $\mathcal{M}_k$ (with cardinality $M_k = |\mathcal{M}_k|$). There is a total number of L sub-carriers denoted by the set $\mathcal{L}$. We consider a frequency reuse-1 system. If a sub-carrier l is allocated to user u by BS k, where $u \in \mathcal{M}_k$, we set $a^l_{k,u} = 1$, and $a^l_{k,u} = 0$ otherwise. Let $p^l_{k,u} \geq 0$ be the power allocated by BS k to user $u \in \mathcal{M}_k$ on sub-carrier l and p^l_k be the total power transmitted by BS k on sub-carrier l, i.e., $p^l_k = \sum_{u\in\mathcal{M}_k} p^l_{k,u}$. We define vector[1]:

[1] First, we fill all the entries of the first BS, followed by the entries of the second BS and continue until the last BS. Among the entries of each BS, first we fill the entries corresponding to the first sub-carrier, followed by the entries of the second sub-carrier and continue until the L-th. Among the entries of each sub-carrier of a BS, we fill the entries corresponding to all its users.

$$\mathbf{a} \triangleq (a^1_{1,1}, \cdots, a^1_{1,M_1}, \cdots, a^2_{1,M_1}, \cdots, \cdots, a^L_{1,M_1}, \cdots, a^1_{K,1}, \cdots, a^1_{K,M_K}, \cdots, a^2_{K,M_K}, \cdots, \cdots, a^L_{K,M_K}).$$

We have $a^l_{k,u} \in \{0,1\}$, for all $k \in \mathcal{K}, u \in \mathcal{M}_k, l \in \mathcal{L}$. We define a vector $\mathbf{p}$ and order the $p^l_{k,u}$, for all $k \in \mathcal{K}, u \in \mathcal{M}_k, l \in \mathcal{L}$, in the same manner. Both $\mathbf{a}$ and $\mathbf{p}$ have length equal to $\sum_{k=1}^K M_k L$.

We consider the use of an SIC based receiver. Such a receiver is characterized by its decoding order of the received signals. The decoding order among the users of BS k on sub-carrier l is defined by the vector $\pi^l_k = (\pi^l_k(1), \pi^l_k(2), \cdots, \pi^l_k(M_k))$, where $\pi^l_k(i)$ is the i-th user to be decoded. In particular, user $\pi^l_k(i)$ is able to decode and subtract the signal of users $\pi^l_k(1)$ to $\pi^l_k(i-1)$ and treats the signal from users $\pi^l_k(i+1)$ to $\pi^l_k(M_k)$ in cell k on sub-carrier l as interference. Notice that there is also inter-cell interference, which should be taken into account when other cells are using the same sub-carrier. The vector π^l_k can thus be seen as a function which maps the decoding order to the user index. Using this formalism, $(\pi^l_k)^{-1}$ is the inverse function, which maps the user index to the corresponding decoding order.

With the above notations, the signal-to-interference-plus-noise ratio (SINR) for a user u served by a BS k on sub-carrier l can be written as:

$$\gamma^l_{k,u} = \frac{g^l_{k,u} p^l_{k,u}}{\sum_{i=(\pi^l_k)^{-1}(u)+1}^{M_k} g^l_{k,u} p^l_{k,\pi^l_k(i)} + \sum_{j\in\mathcal{K}\setminus\{k\}} g^l_{j,u} p^l_j + N^l_{k,u}} \tag{1}$$

where $g^l_{k,u}$ is the link gain between BS k and user u on sub-carrier l, and $N^l_{k,u}$ is the power of the noise for user u in cell k and sub-carrier l. For the simplicity of discussion, in this paper, we would assume that the noise power is constant across users, sub-carriers and BS, i.e., $N^l_{k,u} = N$.

We construct a vector $\boldsymbol{\gamma}$ from the $\gamma^l_{k,u}$ in the same manner as we did for $\mathbf{a}$ and $\mathbf{p}$. We thus have the three following vectors:

$$\mathbf{p} \triangleq [p^l_{k,u}]_{k\in\mathcal{K},l\in\mathcal{L},u\in\mathcal{M}_k}, \tag{2}$$
$$\mathbf{a} \triangleq [a^l_{k,u}]_{k\in\mathcal{K},l\in\mathcal{L},u\in\mathcal{M}_k}, \tag{3}$$
$$\boldsymbol{\gamma} \triangleq [\gamma^l_{k,u}]_{k\in\mathcal{K},l\in\mathcal{L},u\in\mathcal{M}_k}. \tag{4}$$

The maximum transmit power of a BS k on sub-carrier l is denoted by $\bar{p}^l_k$ such that we have the following constraint:

$$0 \leq \sum_{u\in\mathcal{M}_k} p^l_{k,u} \leq \bar{p}^l_k, \forall k \in \mathcal{K}, l \in \mathcal{L}. \tag{5}$$

We denote the maximum total transmit power of a BS k for all the sub-carriers by $\bar{p}_k$, so that we have the following cellular power constraint [3]:

$$0 \leq \sum_{l\in\mathcal{L}} \bar{p}^l_k \leq \bar{p}_k, \forall k \in \mathcal{K}. \tag{6}$$

Because of SIC practical constraints due to decoding complexity and potential error propagation, we consider that there is a limitation on the maximum number of users that we can multiplex in each sub-carrier, denoted by M, i.e.,

$$\sum_{u\in\mathcal{M}_k} a^l_{k,u} \leq M, \forall k \in \mathcal{K}, l \in \mathcal{L}. \tag{7}$$

A. Conditions for Feasible SIC in Multi-cell NOMA

We consider a fixed rule for the SIC decoding order and follow the same SIC ordering as it is done for single cell NOMA[2] in the literature [1]: users are sorted in the increasing order of their link gains; a user with the smallest link gain is decoded first, whereas a user with the largest link gain is decoded last. That is, a weak user (a user with lower link gain) decodes its signal and treats all other signals as interference, while a strong user (with higher link gain) can first decode a weak user's signal to remove it and then decodes its own data.

Nevertheless, a constraint arises on the SIC ordering, which is specific to multi-cell scenario and a condition for SIC to be feasible. Consider the following example, a system with two cells and each cell with two users. Let's call the two users in cell 1 as user 1 and user 2, while the two users in cell 2 as user 3 and user 4. We do our analysis by considering that they are co-channel interferers (says $l = 1$). The power allocated for these users are $p^1_{1,1}$, $p^1_{1,2}$, $p^1_{2,3}$ and $p^1_{2,4}$, respectively. Consider that $g^1_{1,2} > g^1_{1,1}$ and $g^1_{2,4} > g^1_{2,3}$. Before SIC, the SINR for user 1 is $\frac{g^1_{1,1}p^1_{1,1}}{g^1_{1,1}p^1_{1,2}+g^1_{2,1}p^1_2+N}$ and the SINR for user 2 is $\frac{g^1_{1,2}p^1_{1,2}}{g^1_{1,2}p^1_{1,1}+g^1_{2,2}p^1_2+N}$, according to (1). After SIC, the SINR for user 1 (i.e., weak user) would be the same, while the SINR for user 2 (i.e., strong user) would become $\frac{g^1_{1,2}p^1_{1,2}}{g^1_{2,2}p^1_2+N}$ since user 2 can first decode user 1's signal and then remove it. However, note that user 2 can only decode user 1's signal from the received signal if and only if the following condition holds:

$$\frac{g^1_{1,2}p^1_{1,1}}{g^1_{1,2}p^1_{1,2} + g^1_{2,2}p^1_2 + N} > \frac{g^1_{1,1}p^1_{1,1}}{g^1_{1,1}p^1_{1,2} + g^1_{2,1}p^1_2 + N}. \tag{8}$$

By simplifying the above expression, we can obtain $(g^1_{1,2}g^1_{2,1} - g^1_{1,1}g^1_{2,2})p^1_2 + (g^1_{1,2} - g^1_{1,1})N \geq 0$. We state the above observation for general setting as the following theorem.

Theorem 1. *For any two users in $\mathcal{M}_k$ served by BS k on a sub-carrier l such that $g^l_{k,1} < g^l_{k,2}$, a necessary and sufficient condition for user 2 (a strong user) to be able to remove user 1's signal (a weak user) is given by:*

$$\left(\sum_{i\in\mathcal{K}\setminus\{k\}} \left(g^l_{k,2}g^l_{i,1} - g^l_{k,1}g^l_{i,2}\right) p^l_i\right) + \left(g^l_{k,2} - g^l_{k,1}\right) N \geq 0. \tag{9}$$

Proof. If a BS assigns power p to a user, then the achievable rate at the user is given by $\log\left(1 + \frac{g\cdot p}{N_x + I_x}\right)$, where I_x denotes all the interference seen by a user x and N_x is the power of an additive white Gaussian noise at user x. Let us consider two users indexed as 1 and 2, who are served by BS k on sub-carrier l, and $g^l_{k,1} < g^l_{k,2}$. The achievable rate at user 1 is given by $R_1 = \log\left(1 + \frac{g^l_{k,1}p^l_{k,1}}{I_1+N_1}\right)$. Similarly, the achievable rate at user 2 is given by $R_2 = \log\left(1 + \frac{g^l_{k,2}p^l_{k,2}}{I_2+N_2}\right)$. With SIC, denote the SINRs of users 1 and 2 by γ_1 and γ_2 such that

$$\gamma_1 = \frac{g^l_{k,1}p^l_{k,1}}{g^l_{k,1}\sum_{j\in\mathcal{M}_k, j=\pi^l_k(1)+1} p^l_{k,j} + \sum_{i\in\mathcal{K}\setminus\{k\}} g^l_{i,1}p^l_i + N},$$

[2]This means that we do not optimize the SIC ordering for multi-cell. This aspect is known to be an open problem and left for future work.

$$\gamma_2 = \frac{g_{k,2}^l p_{k,1}^l}{g_{k,2}^l \sum_{j\in\mathcal{M}_k, j=\pi_k^l(1)+1} p_{k,j}^l + \sum_{i\in\mathcal{K}\backslash\{k\}} g_{i,2}^l p_i^l + N}.$$

As the SIC order is as such that user 2 should be able to decode the data of user 1, γ_2 has to be greater than or equal to γ_1, i.e., $\gamma_2 - \gamma_1 > 0$, which leads to (9) after simplification. $\square$

Following the result of Theorem 1, we can see that for sub-carrier l and BS k, there are $M_k - 1$ corresponding constraints (necessary and sufficient condition) for SIC to be feasible and hence a total of $\sum_{k=1}^{K} L(M_k - 1)$ constraints for a multi-cell multi-carrier NOMA system.

Consider that $\mathbf{p}_1$ and $\mathbf{p}_2$ are two power vectors as in (2). If $\mathbf{p}_1$ satisfies (9), then any vector $\mathbf{p}_2$, which is coordinate-wise lesser than $\mathbf{p}_1$, may not satisfy (9). The second term in (9) is always non-negative since N is positive and $g_{k,2}^l \geq g_{k,1}^l$, with the chosen decoding order. If $(g_{k,2}^l g_{i,1}^l - g_{k,1}^l g_{i,2}^l) \geq 0, \forall i \in \mathcal{K}\backslash\{k\}$, then every $\mathbf{p} \geq \mathbf{0}$ satisfies (9). Note that $\mathbf{0}$ is a zero vector of the same size of $\mathbf{p}$ and the inequality is coordinate-wise. In Section IV, we conduct a simulation to show that the assumption $(g_{k,2}^l g_{i,1}^l - g_{k,1}^l g_{i,2}^l) \geq 0$ is generally true with high probability in practical reference scenarios.

B. Problem Formulation

The optimization problem is formulated as follows:

$$\max_{\mathbf{a},\mathbf{p}} \quad \sum_{k\in\mathcal{K}} \sum_{u\in\mathcal{M}_k} \sum_{l\in\mathcal{L}} a_{k,u}^l \log(1+\gamma_{k,u}^l) \tag{10}$$

$$\text{subject to} \quad (5), (6), (7), (9)$$

where the constraint (5) ensures that powers are non-negative and the per sub-carrier power constraint is met, (6) is the per BS power constraint, and constraints (7) and (9) are due to SIC. The objective function (10) is the sum rate for multi-cell NOMA. Since $a_{k,u}^l \in \{0,1\}$, we can also write (10) as:

$$\max_{\mathbf{a},\mathbf{p}} \quad \log \prod_{k\in\mathcal{K}} \prod_{u\in\mathcal{M}_k} \prod_{l\in\mathcal{L}} (1 + a_{k,u}^l \gamma_{k,u}^l) \tag{11}$$

$$\text{subject to} \quad (5), (6), (7), (9).$$

III. Optimal Power and Sub-carrier Allocation

In this section, we propose an algorithm to solve the above problem. We start with the preliminary definitions and results.

Definition 1 (Normal Set)**.** *A set $\mathcal{A} \subset \mathbb{R}_+^N$ is a normal set if for any $\mathbf{x} \in \mathcal{A}$, $\{\mathbf{x}' \in \mathbb{R}_+^N | \mathbf{x}' \leq \mathbf{x}\} \subset \mathcal{A}$, where the inequality is component-wise.*

Note that the intersection and the union of normal sets are normal.

Definition 2 (Box)**.** *Given any vector $\mathbf{x} \in \mathbb{R}_+^n$, the hyper rectangle $[\mathbf{0}, \mathbf{x}] = \{\mathbf{v} | \mathbf{0} \leq \mathbf{v} \leq \mathbf{x}\}$ is called a box.*

Note that a box is a normal set.

Definition 3 (Monotonic Optimization)**.** *A monotonic optimization problem is a class of optimization problems:*

$$\max_{\mathbf{x}} \quad f(\mathbf{x}) \tag{12}$$

$$\textit{subject to } \mathbf{x} \in \mathcal{A}$$

where f is an increasing function, $\mathcal{A}$ is a normal set.

A. Monotonic Optimization

Following the approach proposed in [5], we replace the expression $(1 + a_{k,u}^l \gamma_{k,u}^l)$ in (11) by a new variable $z_{k,u}^l$ and re-write (11) as:

$$\max_{\mathbf{z}} \quad \log \prod_{k\in\mathcal{K}} \prod_{u\in\mathcal{M}_k} \prod_{l\in\mathcal{L}} z_{k,u}^l \tag{13}$$

$$\text{subject to} \quad \mathbf{z} \in \mathcal{Z}$$

where $\mathbf{z}$ is the vector that comprises all the $z_{k,u}^l$. Sometimes, we may simply write z_i for its i-th component, where $i = 1, 2, \ldots, MKL$. We call this vector as the SINR vector. Let $\mathcal{Z}$ be the set of all possible $\mathbf{z}$. Let $\mathbf{z}^*$ be an SINR vector solution of (13), and $\mathbf{a}^*$ and $\mathbf{p}^*$ be the corresponding power and SINR vectors respectively. Let N_1 be the number of components of $\mathbf{z}^*$ which take the value greater than 1, where $0 \leq N_1 \leq MKL$. When a component $\mathbf{z}^*$ is 1, the corresponding components of $\mathbf{a}^*$ and $\mathbf{p}^*$ take the value 0. When a component of $\mathbf{z}^*$ is greater than 1, the corresponding component of $\mathbf{a}^*$ would take the value 1. We rewrite $1 + \gamma_{k,u}^l$ as $\frac{f_i(\mathbf{p})}{g_i(\mathbf{p})}$, where f_i and g_i represent the linear functions of $\mathbf{p}$ corresponding to the i-th component of $\mathbf{z}$. Thus, we get N_1 linear equations of the form $z_i g_i - f_i = 0$, $\forall i$, such that $z_i > 1$. The coefficients of g_i and f_i are all independent random channel gains and we can show that with probability 1, all N_1 linear equations are linearly independent, implying that there is a unique $\mathbf{p}$ for every $\mathbf{z}$. The corresponding $\mathbf{p}$ for every $\mathbf{z} \in \mathcal{Z}$ must satisfy (5), (6), (7), (9). Thus, from the optimal solution $\mathbf{z}^*$ of (13), we get the optimal user allocation $\mathbf{a}^*$ and optimal power allocation $\mathbf{p}^*$. We state and prove the following lemma.

Lemma 1. *If there exists two SINR vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ such that $\mathbf{z}_1 \leq \mathbf{z}_2$, then the corresponding power vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ must satisfy the relation $\mathbf{p}_1 \leq \mathbf{p}_2$.*

Proof. Let $\mathcal{I}$ be the set which stores those indices of $\mathbf{z}_2$ where it takes the value 1, i.e., $i \in \mathcal{I}$ if $z_{2i} = 1$. As, $\mathbf{z}_2 \geq \mathbf{z}_1$, the elements of $\mathbf{z}_1$ corresponding to the elements of $\mathcal{I}$ also take the value 1, i.e., $z_{1i} = 1, \forall i \in \mathcal{I}$. Thus, the corresponding elements of both $\mathbf{p}_1$ and $\mathbf{p}_2$ take the value 0 and for those elements the statement of lemma holds true. Let $\mathcal{I}'$ be the set which stores those indices of $\mathbf{z}_2$ where it take the value greater than 1, i.e., $i \in \mathcal{I}'$ if $z_{2i} > 1$. In the following, we prove the statement of the lemma for these elements by contradiction. Let us assume that $\mathcal{I}''$ be the set which stores the indices for which $p_{1i} \geq p_{2i}$ holds true. We define a real number a and an integer α, which are $\max_{i\in\mathcal{I}''} \frac{p_{1i}}{p_{2i}}$ and $\arg\max_{i\in\mathcal{I}''} \frac{p_{1i}}{p_{2i}}$, respectively. Now, let us assume that $\mathcal{I}'''$ be the set which consists of all the indices of power vectors, for which the corresponding elements of the power vectors appear in the denominator of the α-th element of SINR vector. We define another real number b which is $\max_{i\in\mathcal{I}'''} \frac{p_{1i}}{p_{2i}}$. Note that $a \geq b$. From the definition of z_i, we know $z_i = 1 + \gamma_i$. We define $int_i^{\mathbf{P}}$ as the summation of all interference signals term in γ_i and $sig_i^{\mathbf{P}}$ as the desired signal term in γ_i corresponding to power vector $\mathbf{p}$. We consider both the case of $b < 1$ and the case of $b \geq 1$. When $b < 1$,

$$z_{1\alpha} = 1 + \frac{sig_\alpha^{\mathbf{p}_1}}{int_\alpha^{\mathbf{p}_1} + N} \overset{1}{\geq} 1 + \frac{a\ sig_\alpha^{\mathbf{p}_2}}{b\ int_\alpha^{\mathbf{p}_2} + N} \overset{2}{\geq} 1 + \frac{sig_\alpha^{\mathbf{p}_2}}{int_\alpha^{\mathbf{p}_2} + N} = z_{2\alpha} \quad (14)$$

which contradicts the assumption that $\mathbf{z}_2 \geq \mathbf{z}_1$. A linear combination of all the elements of $\mathbf{p}_2$ corresponding to all the elements of $\mathcal{I}'''$ is present at $int_\alpha^{\mathbf{p}_2}$ and from the definition of b the first inequality holds true. The second inequality is true since $\alpha \geq 1$ and $b < 1$. Now, we consider the case $b \geq 1$,

$$z_{1\alpha} = 1 + \frac{sig_\alpha^{\mathbf{p}_1}}{int_\alpha^{\mathbf{p}_1} + N} \overset{1}{\geq} 1 + \frac{a\ sig_\alpha^{\mathbf{p}_2}}{b\ int_\alpha^{\mathbf{p}_2} + N} \overset{2}{\geq} 1 + \frac{a\ sig_\alpha^{\mathbf{p}_2}}{b(\ int_\alpha^{\mathbf{p}_2} + N)} \overset{3}{\geq} 1 + \frac{sig_\alpha^{\mathbf{p}_2}}{int_\alpha^{\mathbf{p}_2} + N} = z_{2\alpha}. \quad (15)$$

Again, it contradicts the assumption of the lemma. The second and the third inequalities hold true as $b \geq 1$ and $\alpha \geq b$. □

Lemma 2. *The set of SINR vectors, $\mathcal{Z}$ corresponding to the power vectors which satisfies, (5), (6), (7), is a normal set.*

Proof. It is to prove that if $\mathbf{z}_1 \in \mathcal{Z}$, then every $\mathbf{z}_2$ which satisfies $\mathbf{z}_2 \leq \mathbf{z}_1$ should also be in $\mathcal{Z}$. We have given an argument before that every power vector has an one-to-one correspondence with every SINR vector. Let us assume that the power vector corresponding to $\mathbf{z}_1$ is $\mathbf{p}_1$ and the power vector corresponding to $\mathbf{z}_2$ is $\mathbf{p}_2$. As $\mathbf{z}_1 \geq \mathbf{z}_2$, from Lemma 1 we can say that $\mathbf{p}_1 \geq \mathbf{p}_2$. Clearly, if $\mathbf{p}_1 \geq 0$ and satisfies (5)–(7), then $\mathbf{p}_2$ also satisfies (5)–(7). Thus, $\mathbf{z}_2$ also is in $\mathcal{Z}$. □

We propose an optimal algorithm for the problem, when (9) holds true for any $\mathbf{p} \geq \mathbf{0}$. As we have seen in the discussion after Theorem 1 and also from simulation result Fig. 1, this is generally true in a practical setting with high probability. Thus, from Lemma 2, we say that $\mathcal{Z}$ is a normal set.

Let us define $f(\mathbf{z}) \triangleq \log \Pi_{k\in\mathcal{K}} \Pi_{u\in\mathcal{M}_k} \Pi_{l\in\mathcal{L}} (z_{k,u}^l)^{w_u^k}$.

Lemma 3. *The function $f(\mathbf{z})$ is Lipschitz continuous.*

Proof. It suffices to show that $f_n(\mathbf{x}) = \log(\Pi_{i=1}^n x_i^{w_i})$, where $\mathbf{x} = (x_i)_{i=1,\dots,n}$ and $x_i \geq 1$, $\forall i$, is Lipschitz continuous. We prove this by mathematical induction.

For $n = 1$, let $x_1 \geq 1$ and $x_2 \geq 1$ be two scalars and a weight $w_1 > 0$. Recall the well-known logarithmic inequality $\log x \leq (x-1)$, for $x \geq 1$. Without loss of generality, assume that $x_1 \geq x_2$. Thus, we have:

$$|f_1(x_1) - f_1(x_2)| = \left|\log\left(\frac{x_1}{x_2}\right)^{w_1}\right| \leq w_1 \left|\frac{x_1}{x_2} - 1\right| = \frac{w_1}{x_2}|x_1 - x_2| \leq w_1|x_2 - x_1|.$$

We now assume that f_{n-1} is Lipschitz continuous with constant $k_{n-1} > 0$. Consider two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^n$, where the coordinates $x_{1,i} > 1$ and $x_{2,i} > 1, \forall i$. Without loss of generality, consider that $x_{1,n} \geq x_{2,n}$. Also, assume that the weights $w_i > 0$, $\forall i$. Using induction hypothesis, we have:

$$|f_n(\mathbf{x}_1) - f_n(\mathbf{x}_2)| = \left|\sum_{i=1}^n \log x_{1,i}^{w_i} - \sum_{i=1}^n \log x_{2,i}^{w_i}\right|$$
$$\leq \left|\sum_{i=1}^{n-1} \log x_{1,i}^{w_i} - \sum_{i=1}^{n-1} \log x_{2,i}^{w_i}\right| + \left|w_n \log \frac{x_{1,n}}{x_{2,n}}\right|$$
$$\leq (k_{n-1}\sum_{i=1}^{n-1} |x_{1,i} - x_{2,i}|) + w_n|x_{1,n} - x_{2,n}| \leq k_n||\mathbf{x}_1 - \mathbf{x}_2||$$

where $k_n = \max(k_{n-1}, w_n)$ and consider L_1 norm in $\mathbb{R}^n$. □

The problem (13) is a monotonic optimization problem as $\mathcal{Z}$ is a normal set and $f(\mathbf{z})$ is an increasing function of $\mathbf{z}$. To solve (13), we employ the outer polyblock algorithm [10]. We solve (13) in two steps. First, we find the optimal sub-carrier allocation then the optimal power allocation by employing the polyblock algorithm. We propose the following theorem which is essential to find the optimal sub-carrier allocation.

Theorem 2. *If the power distribution over all the K BS for the l-th sub-carrier is $(p_1^l, \cdots, p_K^l)$, then the optimal choice is to assign all the power given to each BS to a user with the highest link gain in that sub-carrier, i.e., let $u^\star = \arg\max_u g_{k,u}^l$ (with any tie-breaking rule), then*

$$a_{k,u}^l = \begin{cases} 1 & \text{if } u = u^\star \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

and

$$p_{k,u}^l = \begin{cases} p_k^l & \text{if } u = u^\star \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

Proof. Let us assume one sub-carrier allocation is as such, says the first sub-carrier of the first BS. As aforementioned, there are at most M users per sub-carrier due to the limitation of SIC. Without loss of generality, let us assume that the first user has the highest channel gain and the second user has the second highest channel gain. We use R_1^1 to denote the sum rate in the first sub-carrier of the above first BS such that

$$R_1^1 = \log\Big(1 + \frac{g_{1,1}^1 p_{1,1}^1}{\sum_{i=2}^K g_{i,1}^1 p_i^1 + N}\Big) + \dots + \log\Big(1 + \frac{g_{1,M}^1 p_{1,M}^1}{\sum_{j=1}^{M-1} g_{1,M}^1 p_{1,j}^1 + \sum_{i=2}^K g_{i,2}^1 p_i^1 + N}\Big)$$
$$= \log\Big(\frac{g_{1,1}^1 p_{1,1}^1 + \sum_{i=2}^K g_{i,1}^1 p_i^1 + N}{g_{1,2}^1 p_{1,1}^1 + \sum_{i=2}^K g_{i,2}^1 p_i^1 + N}\Big) + \dots +$$
$$\log\Big(\sum_{j=1}^M g_{1,M}^1 p_{1,j}^1 + \sum_{i=2}^K g_{i,M}^1 p_i^1 + N\Big) - \log\Big(\sum_{i=2}^K g_{i,1}^1 p_i^1 + N\Big)$$
$$= \log\Big(\frac{g_{1,1}^1 p_{1,1}^1 + \sum_{i=2}^K g_{i,1}^1 p_i^1 + N}{g_{1,2}^1 p_{1,1}^1 + \sum_{i=2}^K g_{i,2}^1 p_i^1 + N}\Big) + \dots + \log\Big(g_{1,M}^1 p_1^1 + \sum_{i=2}^K g_{i,2}^1 p_i^1 + N\Big) - \log\Big(\sum_{i=2}^K g_{i,1}^1 p_i^1 + N\Big).$$

We now show that the first term of the above expression is an increasing function of $p_{1,1}^1$. Let us define:

$$f(p_{1,1}^1) \triangleq \log\Big(\frac{g_{1,1}^1 p_{1,1}^1 + \sum_{i=2}^K g_{i,1}^1 p_i^1 + N}{g_{1,2}^1 p_{1,1}^1 + \sum_{i=2}^K g_{i,2}^1 p_i^1 + N}\Big).$$

Taking the derivative of the above function with respect to

$p^1_{1,1}$, we have:

$$\frac{d(f(p^1_{1,1}))}{dp^1_{1,1}} = \frac{g^1_{1,1}(g^1_{1,2}p^1_{1,1} + \sum_{i=2}^{K} g^1_{i,2}p^1_i + N) - g^1_{1,2}(g^1_{1,1}p^1_{1,1} + \sum_{i=2}^{K} g^1_{i,1}p^1_i + N)}{(g^1_{1,1}p^1_{1,1} + \sum_{i=2}^{K} g^1_{i,1}p^1_i + N) \times (g^1_{1,2}p^1_{1,1} + \sum_{i=2}^{K} g^1_{i,2}p^1_i + N)}$$

which is always greater than zero because of the SIC condition (9). We can do the same analysis and reach the same conclusion for all the other terms except the last and the second last terms. For a given $\mathbf{p}$, the second last and the last terms are constant. Thus, at each particular iteration of the polyblock algorithm, if the power distribution over all the BS for a particular sub-carrier l is $(p^l_1, \ldots, p^l_K)$, the optimal choice is to allocate all the power to the best user in that sub-carrier. □

Using Theorem 2, we can get the optimal sub-carrier allocation. In each sub-carrier and in each BS, we choose the user with the best channel gain to get the optimal sub-carrier allocation $\mathbf{a}^*$. To obtain the coordinates of $\mathbf{a}^*$, which correspond to the best channel gain user in each sub-carrier and in each BS, we put 1, while to the other coordinates, we put 0.

Now, we apply the polyblock algorithm to get an optimal power allocation $\mathbf{p}^*$ (see Algorithm 1). We first create an initial SINR vector (see line 3), where the maximum sub-carrier power is allocated to the active user in the sub-carrier allocation while interference is ignored. This vector is clearly infeasible and a box defined by this vertex covers the feasible set of SINR vectors defined by power constraints (5) and (6). We project (see line 8) this vector on the feasible set using Dinkelbach algorithm [11] (see Algorithm 2). From the projected vector, we construct new SINR vectors (see line 9) and then remove the parent vector (see line 10). We repeat the above procedure until the objective value of the projected is ϵ-close to the objective value of the projected vector (see line 15), where ϵ is an input parameter of the algorithm. Since the function $f(\mathbf{z})$ is a Lipschitz continuous function (see Lemma 3), the polyblock algorithm converges to an ϵ-optimal solution in a finite number of iterations [12].

The projection of $\mathbf{z}$ in Algorithm 1 (see line 8) is performed by the Dinkelbach algorithm (see Algorithm 2). From a feasible power vector $\mathbf{p}$, we define a vector $\mathbf{r}$ of length MKL, whose components are fractions of the allocated powers:

$$r_i = n_i/d_i, \forall i, \tag{18}$$

where

$$n_i = g^l_{k,u}p^l_{k,u} + g^l_{k,u} \textstyle\sum_{j \in \mathcal{M}_k, j=\pi^l_k(u)+1} p^l_{k,j} + \sum_{i \in \mathcal{K}\setminus\{k\}} g^l_{i,u}p^l_i + N, \tag{19}$$

$$d_i = g^l_{k,u} \textstyle\sum_{j \in \mathcal{M}_k, j=\pi^l_k(u)+1} p^l_{k,j} + \sum_{i \in \mathcal{K}\setminus\{k\}} g^l_{i,u}p^l_i + N, \tag{20}$$

and i is mapped to the triplet (k, u, l) corresponding to BS k, user u and sub-carrier l.

Algorithm 1 Polyblock approximation algorithm

1: **Inputs**: ϵ, $g^l_{k,u}$, $\bar{p}_k$, $\bar{p}^l_k$, $\forall l \in \mathcal{L}$, $\forall u \in \mathcal{M}_k$, $\forall k \in \mathcal{K}$, N, M
2: **Define**: $f^\star \leftarrow -\infty$
3: **Initialization** : For the optimal sub-carrier allocation $\mathbf{a}^*$, define an SINR vector $\mathbf{z}_{\mathbf{a}^*}$ such that $z^l_{k,u} \leftarrow 1 + g^l_{k,u}\bar{p}_k/N$ if $a^l_{k,u} = 1$ and $z^l_{k,u} \leftarrow 0$ otherwise.
4: $\mathcal{V} \leftarrow \{\mathbf{z}_{\mathbf{a}^*}\}$
5: $f^\star \leftarrow -\infty$
6: **repeat**
7: $\quad \mathbf{z} \leftarrow \arg\max_{\mathbf{z_a} \in \mathcal{V}} f(\mathbf{z_a})$
8: $\quad [\pi_z(\mathbf{z}), \lambda] \leftarrow$ Algorithm 2
9: $\quad \mathbf{z}_i \leftarrow \mathbf{z} - (\mathbf{z} - \pi_z(\mathbf{z})) \circ \mathbf{e}_i$, $\forall i$, where $\mathbf{e}_i$ is the i-th basis vector, $\circ$ is the point-wise multiplication.
10: $\quad \mathcal{V} \leftarrow \mathcal{V} - \{\mathbf{z}\} \cup_i \mathbf{z}_i$, $\forall i$, subject to $a_i = 1$, $p_i > 0$
11: $\quad$ **if** $f(\pi_z(\mathbf{z})) \geq f^\star$ **then**
12: $\quad\quad f^\star \leftarrow f(\pi_z(\mathbf{z}))$
13: $\quad\quad \mathbf{z}^\star \leftarrow \pi_z(\mathbf{z})$
14: $\quad$ **end if**
15: **until** $|f(\mathbf{z}) - f^\star| \leq \epsilon$
16: **Output:** $\mathbf{z}^\star$

Algorithm 2 Dinkelbach algorithm

1: **Inputs**: $\mathbf{z}$, ϵ, $g^l_{k,u}$, $\bar{p}_k$, $\bar{p}^l_k$, $\forall l \in \mathcal{L}$, $\forall u \in \mathcal{M}_k$, $\forall k \in \mathcal{K}$, N, M
2: **Initialization**: $\mathbf{p} \leftarrow \mathbf{0}$
3: Compute $\mathbf{r}$ corresponding to $\mathbf{p}$, using (19) and (20)
4: **repeat**
5: $\quad \lambda \leftarrow \min_i \left(\frac{n_i}{d_i z_i}\right)$
6: $\quad \mathbf{p} \leftarrow \arg\max_{\mathbf{p} \in \mathcal{P}} \min_i n_i - \lambda d_i z_i$
7: $\quad$ Compute $\mathbf{r}$ and $\mathbf{z}$ using $\mathbf{p}$
8: **until** $\min_i n_i - \lambda d_i z_i \geq 0$
9: **Output:**$[\lambda\mathbf{z}, \lambda]$

IV. Simulation Result

In our numerical studies, we consider two neighboring BS, with each BS having 2 sub-carriers and 3 users. The SIC system constraint parameter M in (7) is set to 2. We consider hexagonal cell of radius equal to 100 meters and the users are dropped in each cell randomly following a uniform distribution. We follow the radio propagation model of [13] with distance-dependent path loss $128.1 + 37.6\log_{10} d$, where d is the distance between a BS and the user. We consider that the sub-carrier bandwidth equals to 1 MHz while the noise spectral density is -174 dBm/Hz.

To begin with, we show the empirical cumulative distribution function (CDF) of the quantity $(g^l_{k,2}g^l_{i,1} - g^l_{k,1}g^l_{i,2})$ discussed in Theorem 1, for cell radii of 100, 200 and 500 meters. As shown in Fig. 1, we can see that the probability that $(g^l_{k,2}g^l_{i,1} - g^l_{k,1}g^l_{i,2}) \geq 0$ is very high for these reference scenarios. Thus, we can suggest that with very high probability the constraint (9) can hold for every $\mathbf{p} \geq \mathbf{0}$.

In Fig. 2, we show the sum rate obtained by the proposed algorithm with ϵ set to 0.1, 0.5 and 1, respectively. Meanwhile, we vary the maximum transmit power per sub-carrier value $\bar{p}^l_k$. Besides, we compare with the algorithm proposed in [9], which is a state-of-the-art heuristic optimization algorithm for multi-cell NOMA system. Result shows that the proposed algorithm can outperform the reference algorithm [9] and

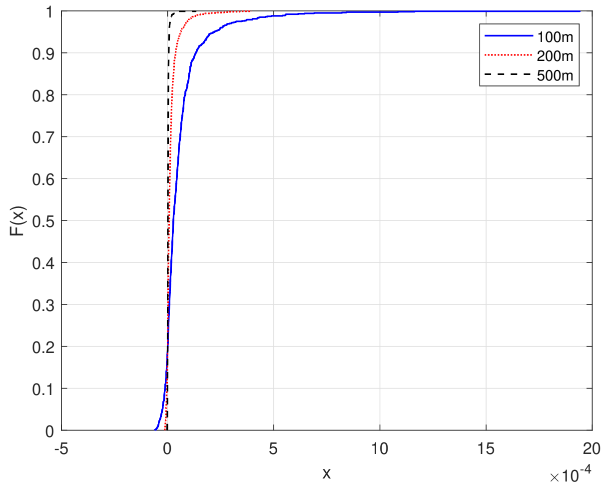

Fig. 1. CDF plot of $(g_{1,2}^l g_{2,1}^l - g_{1,1}^l g_{1,2}^l)$ with cell radius equal to 100, 200 and 500 meters, respectively.

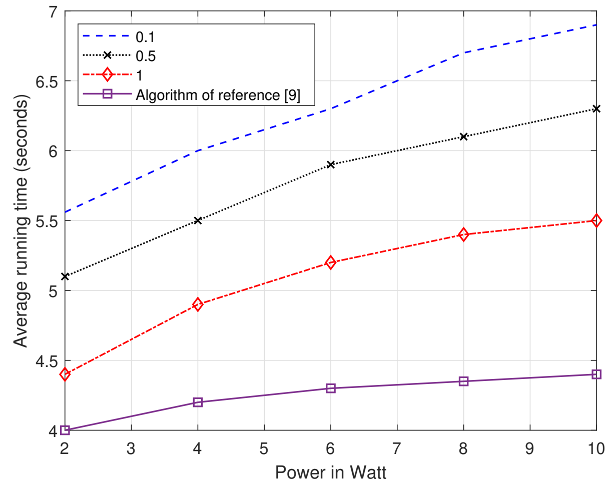

Fig. 3. Average running time of the proposed algorithm in comparison to that of reference [9].

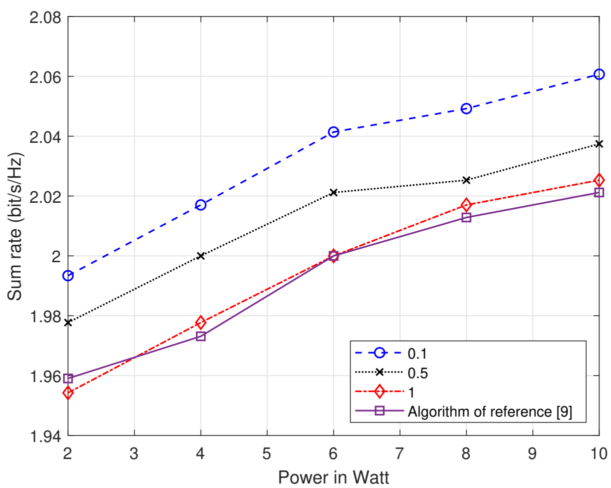

Fig. 2. Sum rate evaluation of NOMA system with varying maximum transmit power per sub-carrier.

provide more optimal solution.

In Fig. 3, we plot the average running time taken by the proposed algorithm in parallel to that taken by the algorithm of [9] to compare their computational complexity.[3] As expected, with a larger ϵ (which is the approximation parameter), the time taken by the proposed algorithm would be shorter and is getting closer to the time taken by the algorithm of [9]. However, we can see that for $\epsilon = 0.1$ to 1, overall the time complexity of the proposed algorithm is comparable to that of [9]. Result shows that it would be suitable to set the proposed algorithm with a small ϵ such as 0.1 at the reasonable time cost for finding a more optimal solution to the sum rate maximization problem.

V. Conclusion

In this paper, we have proposed a global optimal algorithm for the joint power and sub-carrier allocation under multi-cell NOMA systems. Our scheme is based on the polyblock algorithm. It has reduced complexity due to the obtained necessary and sufficient condition for feasible SIC. In practice, we can further adjust the approximation parameter to serve as benchmark solution or to provide suitable solution for solving the multi-cell multi-carrier NOMA resource allocation problem. Simulation result and the comparative study have also shown the effectiveness of the proposed scheme.

[3] It is run on a laptop with Windows 10, 64-bit OS, Intel i5-4590 CPU with 16 GB of RAM. In practice, one can execute on more powerful computers.

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