Weighted Sum-Rate Maximization in Multi-Carrier NOMA with Cellular Power Constraint

Abstract—**Non-orthogonal multiple access (NOMA) has received significant attention for future wireless networks. NOMA outperforms orthogonal schemes, such as OFDMA, in terms of spectral efficiency and massive connectivity. The joint subcarrier and power allocation problem in NOMA is NP-hard to solve in general, due to complex impacts of signal superposition on each user's achievable data rates, as well as combinatorial constraints on the number of multiplexed users per sub-carrier to mitigate error propagation. In this family of problems, weighted sum-rate (WSR) is an important objective function as it can achieve different tradeoffs between sum-rate performance and user fairness. We propose a novel approach to solve the WSR maximization problem in multi-carrier NOMA with cellular power constraint. The problem is divided into two polynomial time solvable sub-problems. First, the multi-carrier power control (given a fixed subcarrier allocation) is non-convex. By taking advantage of its separability property, we design an optimal and low complexity algorithm (MCPC) based on projected gradient descent. Secondly, the single-carrier user selection is a non-convex mixed-integer problem that we solve using dynamic programming (SCUS). This work also aims to give an understanding on how each sub-problem's particular structure can facilitate the algorithm design. In that respect, the above MCPC and SCUS are basic building blocks that can be applied in a wide range of resource allocation problems. Furthermore, we propose an efficient heuristic to solve the general WSR maximization problem by combining MCPC and SCUS. Numerical results show that it achieves near-optimal sum-rate with user fairness, as well as significant performance improvement over OMA.**

I. Introduction

In multi-carrier multiple access systems, the total frequency bandwidth is divided into subcarriers and assigned to users to optimize the spectrum utilization. Orthogonal multiple access (OMA), such as orthogonal frequency-division multiple access (OFDMA) adopted in 3GPP-Long Term Evolution 4G standards [1], only serves one user per subcarrier in order to avoid intra-cell interference and have low-complexity signal decoding at the receiver side. OMA is known to be suboptimal in terms of spectral efficiency [2].

The principle of multi-carrier non-orthogonal multiple access (MC-NOMA) is to multiplex several users on the same subcarrier by performing signals superposition at the transmitter side. Successive interference cancellation (SIC) is applied at the receiver side to mitigate interference between superposed signals. MC-NOMA is a promising multiple access technology for 5G mobile networks as it achieves better spectral efficiency and higher data rates than OMA schemes [3]–[5].

Careful optimization of the transmit powers is required to control the intra-carrier interference of superposed signals and maximize the achievable data rates. Besides, due to error propagation and decoding complexity concerns in practice [6], subcarrier allocation for each transmission also needs to be optimized. As a consequence, joint subcarrier and power allocation problems in NOMA have received much attention. In this class of problems, weighted sum-rate (WSR) maximization is especially important as it can achieve different tradeoffs between sum-rate performance and user fairness [7]. One may also consider to use the weights to perform fair resource allocation in a stable scheduling [8], [9].

Two types of power constraints are considered in the literature. On the one hand, cellular power constraint is mostly used in downlink transmissions to represent the total transmit power budget available at the base station (BS). On the other hand, individual power constraint sets a power limit independently for each user. The latter is clearly more appropriate for uplink scenarios [10], [11], nevertheless it can also be applied to the downlink [12].

It is known that WSR maximization in OFDMA systems is polynomial time solvable if we consider cellular power constraint [13] but strongly NP-hard with individual power constraints [14]. It has also been proven that MC-NOMA with individual power constraints is strongly NP-hard [12]. Several algorithms are developed to perform subcarrier and/or power allocation in this setting. Fractional transmit power control (FTPC) is a simple heuristic that allocates a fraction of the total power budget to each user based on their channel condition [15]. In [10] and [11], heuristic user pairing strategies and iterative resource allocation algorithms are studied for uplink transmissions. A time efficient iterative waterfilling heuristic is introduced in [16] to solve the problem with equal weights. Reference [12] derives an upper bound on the optimal WSR and proposes a near-optimal scheme using dynamic programming and Lagrangian duality techniques. This scheme serves as benchmark due to its high computational complexity, which may not be suitable for practical systems with low latency requirements.

To the best of our knowledge, it is not known if MC-NOMA with cellular power constraint is polynomial time solvable or NP-hard. Only a few papers have developed optimization schemes for this problem, which are either heuristics with no theoretical performance guarantee or algorithms with impractical computational complexity. For example, a greedy user selection and heuristic power allocation scheme based on difference-of-convex programming is proposed in [17]. The authors of [18] employ monotonic optimization to develop an optimal resource allocation policy, which serves as benchmark due to its exponential complexity. The algorithm of [12] can also be applied to cellular power constraint scenarios, but it has very high complexity as well.

Motivated by this observation, we investigate the WSR maximization problem in a downlink multi-carrier NOMA system with cellular power constraint. Our contributions are as follows:

1) We propose a novel framework in which the original problem is divided into two polynomial time solvable sub-problems:
   - The multi-carrier power control (given a fixed subcarrier allocation) sub-problem is non-convex. By taking advantage of its separability property, we design an optimal and low complexity algorithm (MCPC) based on projected gradient descent.
   - The single-carrier user selection sub-problem is a non-convex mixed-integer problem that we solve using dynamic programming (SCUS).

   This work also aims to give an understanding on how each sub-problem's particular structure can facilitate the algorithm design. In that respect, the above MCPC and SCUS are basic building blocks that can be applied in a wide range of resource allocation problems. Our analysis of the separability property in MCPC and the combinatorial constraints in SCUS provide mathematical tools which can be used to study other similar resource allocation optimization problems.
2) Papers [17], [18] restrict the number of multiplexed users per subcarrier, denoted by M, to be equal to 2. However, our result can be applied for an arbitrary number of superposed signals on each subcarrier, i.e., $M \geq 1$. The value of M can be set depending on error propagation and decoding complexity concerns in practice.
3) We propose an efficient heuristic to solve the general WSR maximization problem by combining MCPC and SCUS. Numerical results show that it achieves near-optimal sum-rate and user fairness, as well as significant improvement in performance over OMA.

The paper is organized as follows. In Section II, we present the system model and notations. Section III formulates the WSR problem. We introduce our algorithms in Section IV and analyze their optimality and complexity. We show in Section V some numerical results, highlighting our solution's sum-rate and fairness performance, as well as their computational complexity. Finally, we conclude in Section VI.

II. System Model and Notations

We define in this section the system model and notations used throughout the paper. We consider a downlink multi-carrier NOMA system composed of one base station (BS) serving K users. We denote the index set of users by $\mathcal{K} \triangleq \{1, \dots, K\}$, and the set of subcarriers by $\mathcal{N} \triangleq \{1, \dots, N\}$. The total system bandwidth W is divided into N subcarriers of bandwidth $W_n = W/N$, for each $n \in \mathcal{N}$. We assume orthogonal frequency division, so that adjacent subcarriers do not interfere each other. Moreover, each subcarrier $n \in \mathcal{N}$ experiences frequency-flat block fading on its bandwidth W_n.

Let p_k^n denotes the transmit power from the BS to user $k \in \mathcal{K}$ on subcarrier $n \in \mathcal{N}$. User k is said to be active on subcarrier n if $p_k^n > 0$, and inactive otherwise. In addition, let g_k^n be the channel gain between the BS and user k on subcarrier n, and η_k^n be the received noise power. For simplicity of notations, we define the normalized noise power as $\tilde{\eta}_k^n \triangleq \eta_k^n / g_k^n$. We denote by $\boldsymbol{p} \triangleq (p_k^n)_{k \in \mathcal{K}, n \in \mathcal{N}}$ the vector of all transmit powers, and $\boldsymbol{p}^n \triangleq (p_k^n)_{k \in \mathcal{K}}$ the vector of transmit powers on subcarrier n.

In power domain NOMA, several users are multiplexed on the same subcarrier using superposition coding. A common approach adopted in the literature is to limit the number of superposed signals on each subcarrier to be no more than M. The value of M is meant to characterize practical limitations of SIC due to decoding complexity and error propagation [6]. We represent the set of active users on subcarrier n by $\mathcal{U}_n \triangleq \{k \in \mathcal{K} \colon p_k^n > 0\}$. The aforementioned constraint can then be formulated as $\forall n \in \mathcal{N}$, $|\mathcal{U}_n| \leq M$, where $|\cdot|$ denotes the cardinality of a finite set. Each subcarrier is modeled as a multi-user Gaussian broadcast channel [6] and SIC is applied at the receiver side to mitigate intra-band interference.

The SIC decoding order on subcarrier n is usually defined as a permutation over the active users on n, i.e., $\pi_n \colon \{1, \dots, |\mathcal{U}_n|\} \to \mathcal{U}_n$. However, for ease of reading, we choose to represent it by a permutation function over all users $\mathcal{K}$, i.e., $\pi_n \colon \{1, \dots, K\} \to \mathcal{K}$. These two definitions are equivalent in our model since the Shannon capacity (2) does not depend on the inactive users $k \in \mathcal{K} \setminus \mathcal{U}_n$, for which $p_k^n = 0$. For $i \in \{1, \dots, K\}$, $\pi_n(i)$ returns the i-th decoded user's index. Conversely, user k's decoding order is given by $\pi_n^{-1}(k)$.

In this work, we consider the optimal decoding order studied in [6, Section 6.2]. It consists of decoding users' signals from the highest to the lowest normalized noise power:

$$\tilde{\eta}_{\pi_n(1)}^n \geq \tilde{\eta}_{\pi_n(2)}^n \geq \cdots \geq \tilde{\eta}_{\pi_n(K)}^n. \tag{1}$$

User $\pi_n(i)$ first decodes the signals of users $\pi_n(1)$ to $\pi_n(i-1)$ and subtracts them from the superposed signal before decoding its own signal. Interference from users $\pi_n(j)$ for $j > i$ is treated as noise. The maximum achievable data rate of user k on subcarrier n is given by Shannon capacity:

$$\begin{aligned} R_k^n(\boldsymbol{p}^n) &\triangleq W_n \log_2 \left(1 + \frac{g_k^n p_k^n}{\sum_{j=\pi_n^{-1}(k)+1}^K g_k^n p_{\pi_n(j)}^n + \eta_k^n} \right) \\ &\overset{(a)}{=} W_n \log_2 \left(1 + \frac{p_k^n}{\sum_{j=\pi_n^{-1}(k)+1}^K p_{\pi_n(j)}^n + \tilde{\eta}_k^n} \right), \end{aligned} \tag{2}$$

where equality (a) is obtained after normalizing by g_k^n. We assume perfect SIC, therefore interference from users $\pi_n(j)$ for $j < \pi_n^{-1}(k)$ is completely removed in (2).

III. Problem Formulation

Let $\boldsymbol{w} = \{w_1, \dots, w_K\}$ be a sequence of K positive weights. The main focus of this work is to solve the following joint subcarrier and power allocation optimization problem:

$$
\begin{aligned}
\underset{\boldsymbol{p}}{\text{maximize}} \quad & \sum_{k\in\mathcal{K}} w_k \sum_{n\in\mathcal{N}} R_k^n\left(\boldsymbol{p}^n\right), \\
\text{subject to} \quad & C1: \ \sum_{k\in\mathcal{K}} \sum_{n\in\mathcal{N}} p_k^n \leq P_{max}, \\
& C2: \ \sum_{k\in\mathcal{K}} p_k^n \leq P_{max}^n, \ n\in\mathcal{N}, \\
& C3: \ p_k^n \geq 0, \ k\in\mathcal{K}, \ n\in\mathcal{N}, \\
& C4: \ |\mathcal{U}_n| \leq M, \ n\in\mathcal{N}.
\end{aligned} \qquad (\mathcal{P})
$$

The objective of $\mathcal{P}$ is to maximize the system's WSR. As discussed in Section I, this objective function has received much attention since its weights $\boldsymbol{w}$ can be chosen to achieve different tradeoffs between sum-rate performance and fairness [7]. The weights can also be tuned to perform fair resource allocation in a stable scheduling [8], [9]. Note that $C1$ represents the cellular power constraint, i.e., a total power budget P_{max} at the BS. In $C2$, we set a power limit of P_{max}^n for each subcarrier n. This is a common assumption in multi-carrier systems, e.g., [13], [14]. Constraint $C3$ ensures that the allocated powers remain non-negative. Due to decoding complexity and error propagation in SIC [6], we restrict the maximum number of multiplexed users per subcarrier to M in $C4$.

Let us consider the following change of variables:

$$
\forall n\in\mathcal{N}, \ x_i^n \triangleq \begin{cases} \sum_{j=i}^{K} p_{\pi_n(j)}^n, & \text{if } i\in\{1,\ldots,K\}, \\ 0, & \text{if } i = K+1. \end{cases} \qquad (3)
$$

We define $\boldsymbol{x} \triangleq (x_i^n)_{i\in\{1,\ldots,K\},n\in\mathcal{N}}$ and $\boldsymbol{x}^n \triangleq (x_i^n)_{i\in\{1,\ldots,K\}}$.

Lemma 1 (Equivalent problem $\mathcal{P}'$)**.**
Problem $\mathcal{P}$ is equivalent to problem $\mathcal{P}'$ formulated below:

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\textit{maximize}} \quad & f(\boldsymbol{x}) = \sum_{n\in\mathcal{N}} \sum_{i=1}^{K} f_i^n(x_i^n), \\
\textit{subject to} \quad & C1': \ \sum_{n\in\mathcal{N}} x_1^n \leq P_{max}, \\
& C2': \ x_1^n \leq P_{max}^n, \ n\in\mathcal{N}, \\
& C3': \ x_i^n \geq x_{i+1}^n, \ i\in\{1,\ldots,K\}, \ n\in\mathcal{N}, \\
& C3'': \ x_{K+1}^n = 0, \ n\in\mathcal{N}, \\
& C4': \ |\mathcal{U}_n'| \leq M, \ n\in\mathcal{N},
\end{aligned} \qquad (\mathcal{P}')
$$

where for any $i\in\{1,\ldots,K\}$ and $n\in\mathcal{N}$, we have:

$$
f_i^n(x_i^n) \triangleq \begin{cases} W_n \log_2\left(\left(x_1^n + \tilde{\eta}_{\pi_n(1)}^n\right)^{w_{\pi_n(1)}}\right), & \textit{if } i = 1, \\ W_n \log_2\left(\frac{\left(x_i^n+\tilde{\eta}_{\pi_n(i)}^n\right)^{w_{\pi_n(i)}}}{\left(x_i^n+\tilde{\eta}_{\pi_n(i-1)}^n\right)^{w_{\pi_n(i-1)}}}\right), & \textit{if } i > 1, \end{cases}
$$

and where $\mathcal{U}_n' \triangleq \{i\in\{1,\ldots,K\}\colon x_i^n > x_{i+1}^n\}$.

Proof: See Appendix A. □

The advantage of this formulation $\mathcal{P}'$ is that it exhibits a separable objective function in both dimensions $i\in\{1,\ldots,K\}$ and $n\in\mathcal{N}$. In other words, it can be written as a sum of functions f_i^n, each only depending on one variable x_i^n. In the following section, we take advantage of this separability property to design efficient algorithms to solve $\mathcal{P}'$. The solution of $\mathcal{P}$ can then be obtained by solving $\mathcal{P}'$.

IV. Algorithm Design

Decomposing the joint subcarrier and power allocation problem $\mathcal{P}'$ into smaller sub-problems would allow us to develop efficient algorithms by taking advantage of each sub-problem's particular structure. The technical details are given below.

A. Single-Carrier User Selection

In this subsection, we restrict the optimization search space to a single subcarrier. Given $n\in\mathcal{N}$ and a power budget $\bar{P}^n$, the single-carrier user selection sub-problem $\mathcal{P}_{SCUS}'(n)$ consists in finding a power allocation on n satisfying the multiplexing and SIC constraint $C4'$.

$$
\begin{aligned}
\underset{\boldsymbol{x}^n}{\text{maximize}} \quad & \sum_{i=1}^{K} f_i^n(x_i^n), \\
\text{subject to} \quad & C2', C3', C3'', C4'.
\end{aligned} \qquad (\mathcal{P}_{SCUS}'(n))
$$

Let us introduce auxiliary functions that will help us in our algorithm design. For $n\in\mathcal{N}$, $i\in\{1,\ldots,K\}$ and $j\leq i$, assume that the consecutive variables $x_j^n,\ldots,x_i^n$ are all equal to a certain value $x\in\left[0,\bar{P}^n\right]$. We define $f_{j,i}^n$ as:

$$
\begin{aligned}
f_{j,i}^n(x) &\triangleq \sum_{l=j}^{i} f_l^n(x) \\
&= \begin{cases} W_n \log_2\left(\left(x+\tilde{\eta}_{\pi_n(i)}^n\right)^{w_{\pi_n(i)}}\right), & \text{if } j = 1, \\ W_n \log_2\left(\frac{\left(x+\tilde{\eta}_{\pi_n(i)}^n\right)^{w_{\pi_n(i)}}}{\left(x+\tilde{\eta}_{\pi_n(j-1)}^n\right)^{w_{\pi_n(j-1)}}}\right), & \text{if } j > 1. \end{cases}
\end{aligned}
$$

This simplification of notation is relevant for the analysis of SCUS (Algorithm 2) and also the coming MCPC (Algorithm 4). Indeed, if users $j,\ldots,i-1$ are not active (i.e., $j,\ldots,i-1 \notin \mathcal{U}_n'$), then $x_j^n = \cdots = x_i^n$, therefore $\sum_{l=j}^{i} f_l^n$ can be replaced by $f_{j,i}^n$ and $x_{j+1}^n,\ldots,x_i^n$ are redundant with x_j^n. If constraint $C4'$ is satisfied, up to M users are active on each subcarrier. Thus, evaluating and optimizing the objective function of $\mathcal{P}_{SCUS}'(n)$ only requires $O(M)$ operations.

We study the properties of $f_{j,i}^n$ in Lemma 2. Note that $f_i^n = f_{i,i}^n$, therefore Lemma 2 also holds for functions f_i^n. Fig. 1 shows an example of $f_{j,i}^n$.

Lemma 2 (Properties of $f_{j,i}^n$)**.**
Let $n\in\mathcal{N}$, $i\in\{1,\ldots,K\}$, and $j\leq i$, we have:

- *If $j=1$ or $w_{\pi_n(i)} \geq w_{\pi_n(j-1)}$, then $f_{j,i}^n$ is strictly increasing and concave on $[0,\infty)$.*
- *Otherwise when $j>1$ and $w_{\pi_n(i)} < w_{\pi_n(j-1)}$, $f_{j,i}^n$ is strongly unimodal. It increases on $\left(-\tilde{\eta}_{\pi_n(j-1)}, c_1\right]$ and decreases on $[c_1,\infty)$, where*

$$
c_1 = \frac{w_{\pi_n(j-1)}\tilde{\eta}_{\pi_n(i)} - w_{\pi_n(i)}\tilde{\eta}_{\pi_n(j-1)}}{w_{\pi_n(i)} - w_{\pi_n(j-1)}}.
$$

Besides, $f^n_{j,i}$ *is concave on* $\left(-\tilde{\eta}_{\pi_n(j-1)}, c_2\right]$ *and convex on* $[c_2, \infty)$*, where*

$$c_2 = \frac{\sqrt{w_{\pi_n(j-1)}}\tilde{\eta}_{\pi_n(i)} - \sqrt{w_{\pi_n(i)}}\tilde{\eta}_{\pi_n(j-1)}}{\sqrt{w_{\pi_n(i)}} - \sqrt{w_{\pi_n(j-1)}}} \geq c_1.$$

Proof: The idea consists in studying the first and second derivatives of $f^n_{j,i}$. Details are omitted due to space limitation. □

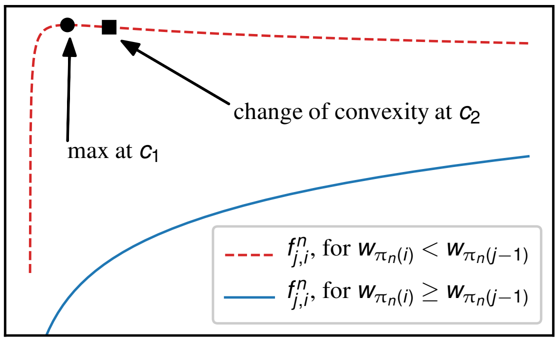

Fig. 1. The two general forms of functions $f^n_{j,i}$

We present in Algorithm 1 the pseudocode MAXF which computes the maximum of $f^n_{j,i}$ on $\left[0, \bar{P}^n\right]$ following the result of Lemma 2. MAXF only requires a constant number of basic operations, therefore its complexity is $O(1)$. For ease of reading, we summarize some system parameters of a given instance of $\mathcal{P}$, for all $n \in \mathcal{N}$, as follows:

$$\mathcal{I}^n = (\boldsymbol{w}, \mathcal{K}, W_n, (g^n_k)_{k\in\mathcal{K}}, (\eta^n_k)_{k\in\mathcal{K}})\,.$$

Algorithm 1 Compute maximum of $f^n_{j,i}$ on $\left[0, \bar{P}^n\right]$

function MAXF$(j, i, \mathcal{I}^n, \bar{P}^n)$
- $a \leftarrow \pi_n(i)$
- $b \leftarrow \pi_n(j-1)$
- **if** $j = 1$ or $w_a \geq w_b$ **then**
  - **return** $\bar{P}^n$
- **else**
  - **return** $\max\left\{0, \min\left\{\frac{w_b\tilde{\eta}_a - w_a\tilde{\eta}_b}{w_a - w_b}, \bar{P}^n\right\}\right\}$
- **end if**

end function

We propose to solve $\mathcal{P}'_{SCUS}(n)$ using Algorithm 2 (SCUS) based on dynamic programming (DP). This approach consists in computing recursively the elements of three arrays V, X, T. Let $m \in \{1, \ldots, M\}$, $j \in \{1, \ldots, K\}$ and $i \geq j$, we define $V[m, j, i]$ as the optimal value of the objective function $\sum_{i=1}^K f^n_i(x^n_i)$ satisfying:

(A) Constraints $C2'$, $C3'$ and $C3''$,
(B) Variables $x^n_j, \ldots, x^n_i$ are all equal, i.e., $x^n_j = \cdots = x^n_i$,
(C) Variables with index $l > i$ are equal to zero, i.e., $x^n_l = 0$,
(D) There are at most m active users, i.e., $\mathcal{U}'_n \leq m$.

$X[m, j, i]$ contains the corresponding optimal value of $x^n_j, \ldots, x^n_i$. Note that this is a scalar, as these variables are all equal from criterion (B). These arrays are computed recursively, i.e., their value at index (m, j, i) depends on their value either at $(m-1, j-1, j-1)$, $(m, j-1, j-1)$ or $(m, j-1, i)$. One of these indexes used in the computation of $V[m, j, i]$ and $X[m, j, i]$, is stored in $U[m, j, i]$ to keep track of the optimal allocation on previous variables x^n_l, $l < j$. A line-by-line description and analysis is given below.

Lines 1 to 6: We initialize the arrays for index (m, j, i) with $j = 1$, $1 \leq i \leq K$ and $m \in \{1, \ldots, M\}$. It is direct from the above definition that $x^n_1 = \cdots = x^n_i$ and $x^n_{i+1} = \cdots = x^n_{K+1} = 0$. We know from Lemma 2 that the maximum is obtained for $x^n_1 = \cdots = x^n_i = \text{MAXF}(1, i, \mathcal{I}^n, \bar{P}^n) = \bar{P}^n$, which explains the assignment at lines 3 and 4. At line 5, $U[m, 1, i]$ gets $(0, 0, 0)$ as there is no previous index.

Lines 7 to 19: We initialize the arrays for $m = 1$ and $2 \leq j \leq i \leq K$. Since $\mathcal{U}'_n \leq m = 1$ and $C2'$, $C3'$, $C3''$ have to be satisfied, the optimal is achieved by $x^n_1 = \cdots = x^n_l = \bar{P}^n$ and $x^n_{l+1} = \cdots = x^n_{K+1} = 0$, for a certain $l \leq i$. Suppose that $l = i$, then the value of $V[m, j, i]$ is given by v_P at line 8. Otherwise $l < i$, and $V[m, j, i]$ is given by v_0 at line 9. The maximum between v_P and v_0 is assigned to $V[m, j, i]$ at lines 10 to 18. The corresponding allocation is stored in $X[m, j, i]$.

Lines 20 to 39: We compute recursively the elements of V, X, U for $j = 2$ to K, $i = j$ to K, and $m = 2$ to M. At index (m, j, i), there are three potential cases to consider:

- Case v_0: Suppose that user i is inactive, we have $x^n_i = x^n_{i+1}$ by definition of $\mathcal{U}'_n$. If follows from (B) and (C) that $x^n_j = \cdots = x^n_{K+1} = 0$. Thus, $V[m, j, i] = V[m, j-1, j-1]$, which is denoted by v_0 at line 24.
- Case v_1: Assume that users i and $j-1$ are both active. Let $x^* = \text{MAXF}(j, i, \mathcal{I}^n, \bar{P}^n)$ as in line 22. Suppose for the sake of contradiction that $x^* \geq x^n_{j-1}$. According to Lemma 2, $f^n_{j,i}$ is increasing in $[0, x^*]$. As a consequence, the optimal satisfying $C3'$ is achieved for $x^n_j = x^n_{j-1}$, which contradicts with the fact that $j-1$ is active, i.e., $x^n_{j-1} > x^n_j$. We derive from $x^n_{j-1} = X[m-1, j-1, j-1]$ that:

$$x^* < X[m-1, j-1, j-1]. \tag{4}$$

Suppose for the sake of contradiction that $x^* = 0$. We know from Lemma 2 that $f^n_{j,i}$ decreases in $[x^*, +\infty)$. Therefore, the optimal satisfying $C3'$ and $C3''$ is achieved for $x^n_i = 0$, which contradicts with the fact that i is active, i.e., $x^n_i > x^n_{i+1} = 0$. We deduce that:

$$x^* > 0. \tag{5}$$

If (4) and (5) are satisfied, then we have $V[m, j, i] = v_1$. v_1 is the sum of the optimal objective with at most $m-1$ active users from indexes 1 to $j-1$, i.e., $V[m-1, j-1, j-1]$, and the individual contribution of the active user i, i.e., $f^n_{j,i}\left(\text{MAXF}(j, i, \mathcal{I}^n, \bar{P}^n)\right) - f^n_{j,i}(0)$.

- Case v_2: If i is active and $j-1$ is inactive, then $x^n_{j-1} = x^n_j$ by definition of $\mathcal{U}'_n$. Therefore, $V[m, j, i]$ is given by $v_2 = V[m, j-1, i]$ at line 25.

The maximum between v_0, v_1 and v_2 is assigned to $V[m, j, i]$, and the corresponding allocation is stored in $X[m, j, i]$. v_1 is only chosen if, in addition, conditions (4) and (5) are true as shown in line 26.

Algorithm 2 Single-carrier user selection algorithm (SCUS)

function $\text{SCUS}(\mathcal{I}^n, \bar{P}^n, M)$
1: $\triangleright$ Initialize arrays V, X, U for $j = 1$ and $m = 1$
2: **for** $m = 1$ to M and $i = 1$ to K **do**
3: $\quad V[m, 1, i] \leftarrow \begin{cases} f^n_{1,K}(\bar{P}^n), & \text{if } i = K, \\ f^n_{1,i}(\bar{P}^n) + f^n_{i+1,K}(0), & \text{else.} \end{cases}$
4: $\quad X[m, 1, i] \leftarrow \bar{P}^n$
5: $\quad U[m, 1, i] \leftarrow (0, 0, 0)$
6: **end for**
7: **for** $j = 2$ to K and $i = j$ to K **do**
8: $\quad v_P \leftarrow f^n_{1,i}(\bar{P}^n) + f^n_{i+1,K}(0)$
9: $\quad v_0 \leftarrow V[1, j-1, j-1]$
10: $\quad$ **if** $v_P \geq v_0$ **then**
11: $\quad\quad V[1, j, i] \leftarrow v_P$
12: $\quad\quad X[1, j, i] \leftarrow \bar{P}^n$
13: $\quad\quad U[1, j, i] \leftarrow (1, 1, i)$
14: $\quad$ **else**
15: $\quad\quad V[1, j, i] \leftarrow v_0$
16: $\quad\quad X[1, j, i] \leftarrow 0$
17: $\quad\quad U[1, j, i] \leftarrow (1, j-1, j-1)$
18: $\quad$ **end if**
19: **end for**
20: $\triangleright$ Compute V, X, U for $m \leq M$ and $j \leq i \leq K$
21: **for** $j = 2$ to K and $i = j$ to K and $m = 2$ to M **do**
22: $\quad x^* \leftarrow \text{MaxF}(j, i, \mathcal{I}^n, \bar{P}^n)$
23: $\quad v_0 \leftarrow V[m, j-1, j-1]$
24: $\quad v_1 \leftarrow V[m-1, j-1, j-1] + f^n_{j,i}(x^*) - f^n_{j,i}(0)$
25: $\quad v_2 \leftarrow V[m, j-1, i]$
26: $\quad$ **if** $0 < x^* < X[m-1, j-1, j-1]$ and $v_1 \geq v_2, v_0$ **then**
27: $\quad\quad V[m, j, i] \leftarrow v_1$
28: $\quad\quad X[m, j, i] \leftarrow x^*$
29: $\quad\quad U[m, j, i] \leftarrow (m-1, j-1, j-1)$
30: $\quad$ **else if** $v_2 \geq v_0$ **then**
31: $\quad\quad V[m, j, i] \leftarrow v_2$
32: $\quad\quad X[m, j, i] \leftarrow X[m, j-1, i]$
33: $\quad\quad U[m, j, i] \leftarrow (m, j-1, i)$
34: $\quad$ **else**
35: $\quad\quad V[m, j, i] \leftarrow v_0$
36: $\quad\quad X[m, j, i] \leftarrow 0$
37: $\quad\quad U[m, j, i] \leftarrow (m, j-1, j-1)$
38: $\quad$ **end if**
39: **end for**
40: $\triangleright$ Retrieve the optimal solution $\boldsymbol{x}^n$
41: $x^n_1, \ldots, x^n_K \leftarrow 0$
42: $(m, j, i) \leftarrow (M, K, K)$
43: **repeat**
44: $\quad x^n_j, \ldots, x^n_i \leftarrow X[m, j, i]$
45: $\quad (m, j, i) \leftarrow U[m, j, i]$
46: **until** $(m, j, i) = (0, 0, 0)$
47: **return** $\boldsymbol{x}^n$
end function

Lines 40 to 47: Finally, $V[M, K, K]$ is by definition the optimal value of $\mathcal{P}'_{SCUS}(n)$. The corresponding optimal allocation $\boldsymbol{x}^n$ is retrieved in lines 43 to 46 by finding recursively the indexes (m, j, i) in U used to construct the solution until reaching $(0, 0, 0)$. At each iteration, $X[m, j, i]$ is assigned to variables $x^n_j, \ldots, x^n_i$.

Theorem 3 (Optimality and complexity of algorithm SCUS). *Given a subcarrier $n \in \mathcal{N}$, a power budget $\bar{P}^n$ and $M \geq 1$, algorithm* SCUS *computes the optimal single-carrier power control and user selection of $\mathcal{P}'_{SCUS}(n)$. Its worst case computational complexity is $O(MK^2)$.*

Proof: See Appendix B. □

B. Multi-Carrier Power Control

The following multi-carrier power control subproblem $\mathcal{P}'_{MCPC}$ consists in maximizing the WSR taking as input a fixed subcarrier allocation $\mathcal{U}'_n$, for all $n \in \mathcal{N}$, so that constraint $C4'$ can be ignored. Since inactive users $k \notin \mathcal{U}_n$ on each subcarrier $n \in \mathcal{N}$ have no contribution on the data rates, i.e., $p^n_k = 0$ and $R^n_k = 0$, we remove them in the definition of $\mathcal{P}$ for the study of this sub-problem. Without loss of generality, we then transform $\mathcal{P}$ to $\mathcal{P}'$ and index the remaining active users on each subcarrier n by $i_n \in \{1_n, \ldots, |\mathcal{U}'_n|_n\}$. We formulate $\mathcal{P}'_{MCPC}$ as a two-stage optimization problem. The first-stage is:

$$\begin{aligned} &\underset{\bar{\boldsymbol{P}}}{\text{maximize}} && \sum_{n \in \mathcal{N}} F^n(\bar{P}^n) && (\mathcal{P}'_{MCPC}) \\ &\text{subject to} && \bar{P}^n \in \mathcal{F}, \end{aligned}$$

where $\bar{P}^n$, for $n \in \mathcal{N}$, are intermediate variables representing each subcarrier's power budget. $\bar{\boldsymbol{P}} \triangleq \left(\bar{P}^1, \ldots, \bar{P}^N\right)$ denotes the power budget vector. The feasible set

$$\mathcal{F} \triangleq \{\bar{\boldsymbol{P}} \colon \sum_{n \in \mathcal{N}} \bar{P}^n \leq P_{max} \text{ and } 0 \leq \bar{P}^n \leq P^n_{max},\ n \in \mathcal{N}\} \tag{6}$$

is chosen to satisfy $C1'$ and $C2'$. $F^n(\bar{P}^n)$ is the optimal value of the second-stage:

$$\begin{aligned} F^n(\bar{P}^n) \quad = \quad & \max_{x^n_{i_n}} && \sum_{i=1}^{|\mathcal{U}'_n|} f^n_{i_n}(x^n_{i_n}) && (\mathcal{P}'_{SCPC}) \\ & \text{subject to} && x^n_{1_n} \leq \bar{P}^n, \\ &&& x^n_{1_n} \geq x^n_{(i+1)_n},\ i \in \{1, \ldots, |\mathcal{U}'_n|\}, \\ &&& x^n_{(|\mathcal{U}'_n|+1)_n} = 0. \end{aligned}$$

The first-stage $\mathcal{P}'_{MCPC}$ consists in optimizing the allocated power budget $\bar{P}^n$ among subcarriers $n \in \mathcal{N}$. While the second-stage $\mathcal{P}'_{SCPC}$ performs power allocation on each subcarrier n, with a given allocation $\mathcal{U}'_n$ and power budget $\bar{P}^n$. Fig. 2 gives an overview of the nested optimization that we design to solve $\mathcal{P}'_{MCPC}$.

We solve the second-stage $\mathcal{P}'_{SCPC}$ using Algorithm 3 (SCPC). The idea is to iterate over variables $x^n_{i_n}$ for $i = 1$ to $|\mathcal{U}'_n|$, and compute their optimal value $x^* = \text{MaxF}(i_n, i_n, \mathcal{I}^n, \bar{P}^n)$ at line 3. If the current allocation satisfies constraint $C3'$, then $x^n_{i_n}$ gets value x^*. Otherwise, the algorithm backtracks at line 6 and finds the highest index $j \in \{2, \ldots, i-1\}$ such that $x^n_{(j-1)_n} \geq \text{MaxF}(j_n, i_n, \mathcal{I}^n, \bar{P}^n)$. Then, variables $x^n_{j_n}, \ldots, x^n_{i_n}$ are set equal to $\text{MaxF}(j_n, i_n, \mathcal{I}^n, \bar{P}^n)$ at line 10. The optimality and complexity of this algorithm are presented in Theorem 4.

Theorem 4 (Optimality and complexity of algorithm SCPC). *Given subcarrier $n \in \mathcal{N}$ and a power budget $\bar{P}^n$, algorithm* SCPC *computes the optimal single-carrier power control. Its worst case computational complexity is $O(|\mathcal{U}'_n|^2)$.*

Proof: See Appendix C. □

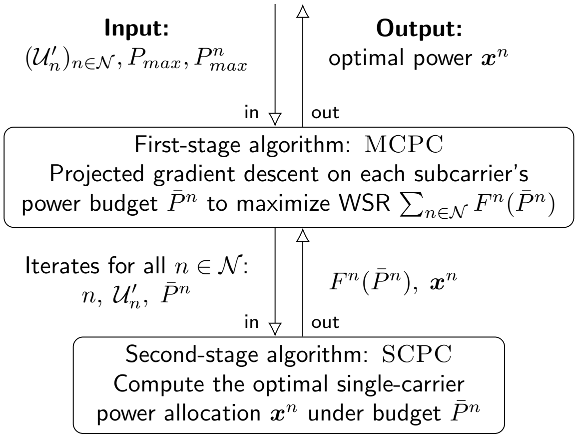

Fig. 2. Overview of MCPC

Algorithm 3 Single-carrier power control algorithm (SCPC)

function SCPC($\mathcal{I}^n, \mathcal{U}'_n, \bar{P}^n$)
  for $i = 1$ to $|\mathcal{U}'_n|$ **do**
    ▷ Compute the optimal of $f^n_{i_n}$
    $x^* \leftarrow$ MAXF($i_n, i_n, \mathcal{I}^n, \bar{P}^n$)
    ▷ Modify x^* if this allocation violates constraint $C3'$
    $j \leftarrow i - 1$
    while $x^n_{j_n} < x^*$ and $j \geq 1$ **do**
      $x^* \leftarrow$ MAXF($j_n, i_n, \mathcal{I}^n, \bar{P}^n$)
      $j \leftarrow j - 1$
    end while
    $x^n_{(j+1)_n}, \ldots, x^n_{i_n} \leftarrow x^*$
  end for
  return $x^n_{1_n}, \ldots, x^n_{|\mathcal{U}'_n|_n}$
end function

Lemma 5 (Convexity of $\mathcal{P}'_{MCPC}$)**.** *$\mathcal{F}$ is a convex set and $\sum_{n\in\mathcal{N}} F^n$ is concave with respect to $\bar{\boldsymbol{P}} \in \mathcal{F}$.*

Proof: See Appendix D. □

According to Lemma 5, $\sum_{n\in\mathcal{N}} F^n$ is concave on the convex feasible set $\mathcal{F}$. Hence, the optimal multi-carrier power allocation $\mathcal{P}'_{MCPC}$ can be computed efficiently using projected gradient descent [19]. This MCPC scheme is described in Algorithm 4, in which the second-stage optimal value $F^n(\bar{P}^n)$, defined in $\mathcal{P}'_{SCPC}$, is determined using SCPC as described in the auxiliary function in lines 12 to 13. The search direction at line 4 can be found either using numerical methods or an exact gradient formula (in the Proof of Lemma 5). Note that the step size α at line 5 can be tuned by backtracking line search or exact line search [19]. We adopt the latter to perform simulations. The projection of $\bar{\boldsymbol{P}} + \alpha\Delta$ on the simplex $\mathcal{F}$ at line 6 can be computed efficiently [20], the details of its implementation are omitted here.

Theorem 6 (Optimality and complexity of algorithm MCPC)**.** *Given a subcarrier allocation such that $|\mathcal{U}'_n| \leq M$, for all $n \in \mathcal{N}$, algorithm* MCPC *converges to the optimal solution of $\mathcal{P}'_{MCPC}$ in $O(\log(1/\epsilon))$ iterations, where ϵ is the error tolerance. Each iteration requires $O(NM^2)$ basic operations.*

Algorithm 4 Multi-carrier power control algorithm (MCPC)

function MCPC$\big((\mathcal{I}^n)_{n\in\mathcal{N}}, (\mathcal{U}'_n)_{n\in\mathcal{N}}, P_{max}, P^n_{max}, \mathcal{N}, \epsilon\big)$
1: Let $\bar{\boldsymbol{P}} \leftarrow \boldsymbol{0}$ be the starting point
2: **repeat**
3:   Save the previous vector $\bar{\boldsymbol{P}}' \leftarrow \bar{\boldsymbol{P}}$
4:   Determine a search direction $\Delta \leftarrow \nabla \sum_{n\in\mathcal{N}} F^n(\bar{P}^n)$
5:   Choose a step size α
6:   Update $\bar{\boldsymbol{P}} \leftarrow$ projection of $\bar{\boldsymbol{P}} + \alpha\Delta$ on $\mathcal{F}$
7: **until** $||\bar{\boldsymbol{P}}' - \bar{\boldsymbol{P}}||_2^2 \leq \epsilon$
8: **for** $n \in \mathcal{N}$ **do**
9:   $x^n_{1_n}, \ldots, x^n_{|\mathcal{U}'_n|_n} \leftarrow$ SCPC $\left(\mathcal{I}^n, \mathcal{U}'_n, \bar{P}^n\right)$
10: **end for**
11: **return** $(x^n_{1_n}, \ldots, x^n_{|\mathcal{U}'_n|_n})_{n\in\mathcal{N}}$
end function

function $F^n\left(\bar{P}^n\right) \triangleq F^n\left(\mathcal{I}^n, \mathcal{U}'_n, \bar{P}^n\right)$
12: $x^n_{1_n}, \ldots, x^n_{|\mathcal{U}'_n|_n} \leftarrow$ SCPC $\left(\mathcal{I}^n, \mathcal{U}'_n, \bar{P}^n\right)$
13: **return** $\sum_{i=1}^{|\mathcal{U}'_n|} f^n_{i_n}(x^n_{i_n})$
end function

Proof: It follows from Lemma 5 and classical convex programming results [19] that projected gradient descent converges to the optimal multi-carrier power control (MCPC) in $O(\log(1/\epsilon))$ iterations. Each iteration requires to compute SCPC($\mathcal{I}^n, \mathcal{U}'_n, \bar{P}^n$), for $n \in \mathcal{N}$. Thus, we derive from Theorem 4 and $|\mathcal{U}'_n| \leq M$ that each iteration's worst case complexity is $O(NM^2)$. □

C. Multi-Carrier Joint Subcarrier and Power Allocation

We design an efficient heuristic joint subcarrier and power allocation scheme, namely JSPA, which combines MCPC and SCUS. We present its principle in Fig. 3. JSPA is similar to the two-stage MCPC depicted in Fig. 2. With the difference that subcarrier allocation is no longer provided as input, and has to be optimized in the second-stage. To this end, we replace the second-stage SCPC by the optimal single-carrier user selection algorithm SCUS studied in Section IV-A. Although SCUS is optimal, the returned $F^n(\bar{P}^n)$ is no longer concave. As a consequence, JSPA is not guaranteed to converge to a global maximum. Each iteration of JSPA requires to compute SCUS($\mathcal{I}^n, \bar{P}^n, M$), for $n \in \mathcal{N}$. Thus, we derive from Theorem 3 that each iteration's worst case complexity is $O(NMK^2)$. Nevertheless, we show by numerical results in the next section that it achieves near-optimal weighted sum-rate performance with low complexity.

V. Numerical Results

We evaluate the WSR, user fairness performance and computational complexity of JSPA through numerical simulations. We compare our proposed scheme with the near-optimal high complexity benchmark scheme LDDP introduced in [12], as well as FTPC with greedy subcarrier allocation, which is also considered in [12] and [15]–[17]. We consider a cell of diameter 500 meters, with one BS located at its center and K users distributed uniformly at random in the cell. The number of users K varies from 5 to 30, and the number of subcarriers is $N = 10$. Each point in the following figures are average

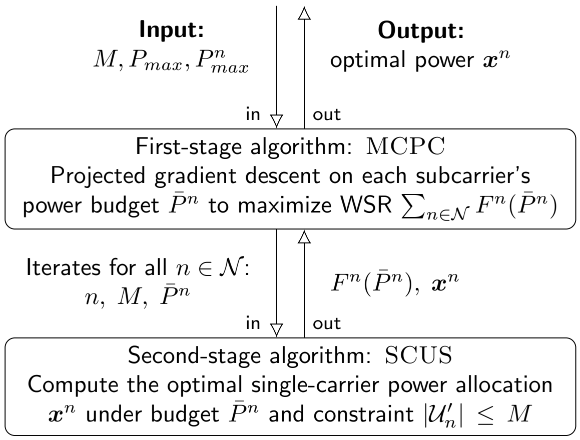

Fig. 3. Overview of JSPA

TABLE I
SIMULATION PARAMETERS

Parameters	Value
Cell radius	250 m
Minimum distance from user to BS	35 m
Carrier frequency	2 GHz
Path loss model	$128.1 + 37.6 \log_{10} d$ dB, d is in km
Shadowing	Log-normal, 8 dB standard deviation
Fading	Rayleigh fading with variance 1
Noise power spectral density	-174 dBm/Hz
System bandwidth W	5 MHz
Number of subcarriers N	10
Number of users K	5 to 30
Total power budget P_{max}	1 W
Error tolerance ϵ	10^{-4}
Parameter M	1 (OMA), 2 and 3 (NOMA)

value obtained over 1000 random instances. The simulation parameters and channel model are summarized in Table I.

Fig. 4 and 5 illustrate the WSR and complexity of JSPA and LDDP for different systems, i.e., OMA with $M = 1$, NOMA with $M = 2$ and $M = 3$. In these two figures, weights $\boldsymbol{w}$ are generated uniformly at random in $[0, 1]$. Algorithm LDDP requires to discretize the total power budget P_{max} into J power levels. Here, we choose $J = 100$. Due to the high computational complexity of LDDP, we only simulate it for $K = 5$ to 20.

In Fig. 4, the performance loss of JSPA compared to LDDP is less than 0.8% for any number of users K. This indicates that JSPA also achieves near-optimal WSR, close to that of LDDP. In addition, we see that the performance gain of NOMA systems compared to OMA is 7% for $M = 2$ and 10% for $M = 3$.

In Fig. 5, we count the number of basic operations (additions, multiplications, comparisons) performed by each scheme, which reflects their computational complexity. As expected from Theorem 4, the complexity of JSPA increases linearly with M (there is a constant difference between the curves of JSPA with $M = 1$, 2 and 3 in the semi-log plot). Our proposed scheme JSPA has low complexity, it runs within seconds on a common computer[1] for $K \leq 30$, whereas LDDP requires 1600 times more operations for $K = 20$ and $M = 2$.

[1]with the following specifications: Python 3, Windows 7, 64bits, AMD A10-5750M APU with Radeon HD Graphics, and 8 GB of RAM.

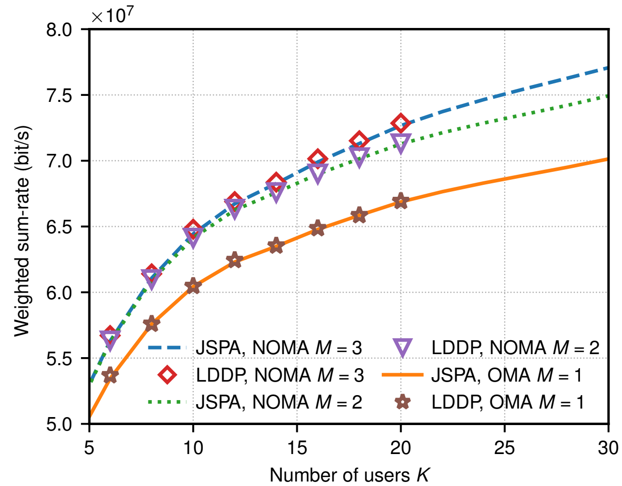

Fig. 4. WSR versus K for JSPA and LDDP

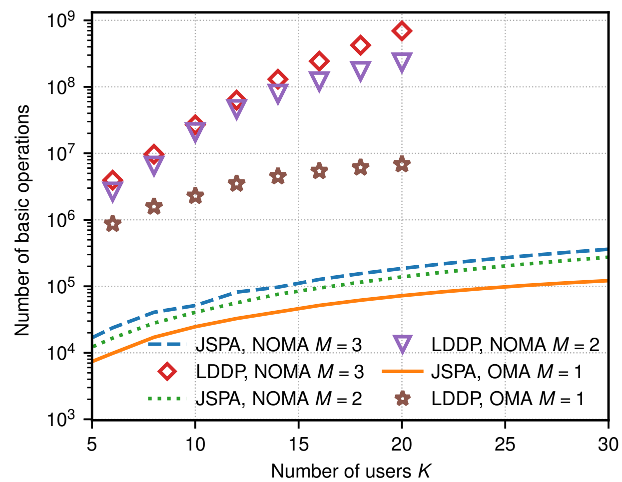

Fig. 5. Number of basic operations versus K for JSPA and LDDP

In Fig. 6 and 7, we implement a proportional fair scheduler on one frame, which is composed of $T = 20$ time slots. The objective of this setup is to evaluate the fairness performance of our scheme when the weights are chosen according to a proportional fair scheduling. The channel state information is collected at the beginning of the frame. Let $R_k(t)$ be user k's data rate at time slot $t \in \{1, \ldots, T\}$, and $\bar{R}_k(t)$ be user k's average data rate prior to t. In the proportional fair scheduling [21], $\bar{R}_k(t)$ is updated as follows:

$$\bar{R}_k(t+1) = \left(1 - \frac{1}{T}\right) \bar{R}_k(t) + \frac{1}{T} R_k(t).$$

The weight of user k at time t is then set as $w_k(t) = 1/\bar{R}_k(t)$, in order to achieve a good tradeoff between spectral efficiency and user fairness [22]. At the end of the frame, we average each user k's data rate over the entire frame, i.e., $R_k^{mean} = \sum_{t=1}^{T} R_k(t)/T$. We show respectively in Fig. 6 and 7, the proportional fairness index defined as $\sum_{k\in\mathcal{K}} \log(R_k^{mean})/K$ and the sum-rate $\sum_{k\in\mathcal{K}} R_k^{mean}$. We compare our scheme JSPA to FTPC with greedy subcarrier allocation. The decay factor of FTPC is set to 0.4, which is a common value in the literature. Our proposed JSPA outperforms the heuristic

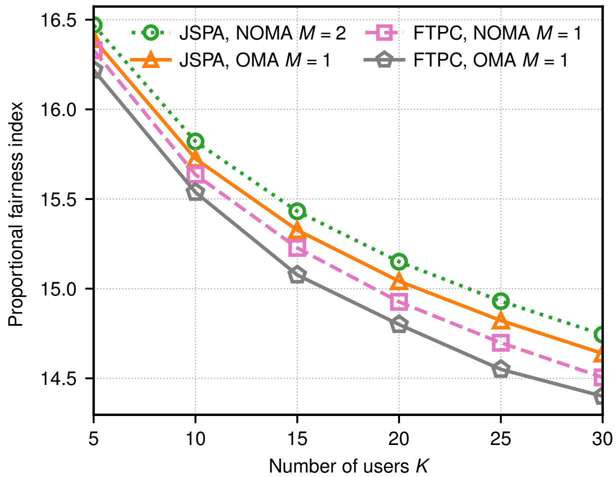

Fig. 6. Proportional fairness index of JSPA and FTPC

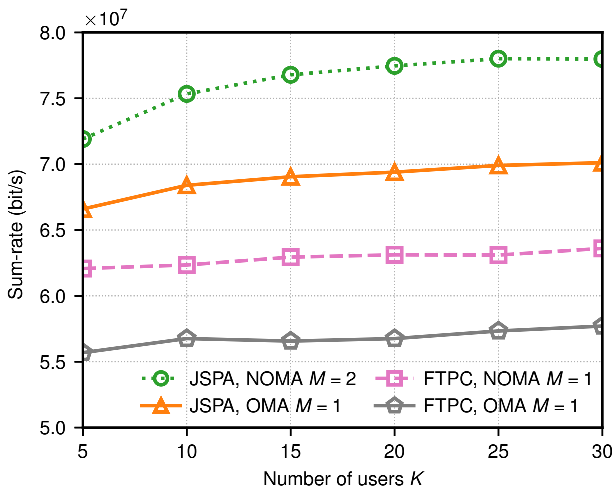

Fig. 7. Sum-rate of JSPA and FTPC in a proportional fair scheduler

FTPC in both user fairness and sum-rate performance, and in both the OMA and NOMA systems. For example, in Fig. 7 for $K = 30$, the sum-rate gain of JSPA in NOMA with $M = 2$ and in OMA are respectively 23% and 21%.

VI. Conclusion

We propose a novel approach to solve the WSR maximization problem in MC-NOMA with cellular power constraint. We prove that two sub-problems can be solved optimally with low complexity. The user selection combinatorial problem is solved by SCUS based on dynamic programming. The multi-carrier power control non-convex problem is solved by the two-stage algorithm MCPC, which uses separability property and gradient descent methods. These algorithms are basic building blocks that can be applied in a wide range of resource allocation problems. Furthermore, we develop an efficient scheme, JSPA, to tackle the joint subcarrier and power allocation problem. We show through numerical results that it achieves near-optimal WSR and user fairness.

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Appendix

A. Proof of Lemma 1

The objective of $\mathcal{P}$ can be written as:

$$\sum_{k\in\mathcal{K}} w_k \sum_{n\in\mathcal{N}} R_k^n\left(\boldsymbol{p}^n\right) = \sum_{n\in\mathcal{N}} \sum_{k\in\mathcal{K}} w_k R_k^n\left(\boldsymbol{p}^n\right)$$

$$\overset{(b)}{=} \sum_{n\in\mathcal{N}} W_n \sum_{i=1}^{K} w_{\pi_n(i)} \log_2\left(\frac{\sum_{j=i}^{K} p^n_{\pi_n(j)} + \tilde{\eta}^n_{\pi_n(i)}}{\sum_{j=i+1}^{K} p^n_{\pi_n(j)} + \tilde{\eta}^n_{\pi_n(i)}}\right)$$

$$\overset{(c)}{=} \sum_{n\in\mathcal{N}} W_n \sum_{i=1}^{K} \log_2 \left(\frac{\left(\sum_{j=i}^{K} p^n_{\pi_n(j)} + \tilde{\eta}^n_{\pi_n(i)}\right)^{w_{\pi_n(i)}}}{\left(\sum_{j=i+1}^{K} p^n_{\pi_n(j)} + \tilde{\eta}^n_{\pi_n(i)}\right)^{w_{\pi_n(i)}}} \right)$$

$$\overset{(d)}{=} \sum_{n\in\mathcal{N}} W_n \Bigg[w_{\pi_n(1)} \log_2 \left(\sum_{j=1}^{K} p^n_{\pi_n(j)} + \tilde{\eta}^n_{\pi_n(1)} \right) + \sum_{i=2}^{K} \log_2 \left(\frac{\left(\sum_{j=i}^{K} p^n_{\pi_n(j)} + \tilde{\eta}^n_{\pi_n(i)}\right)^{w_{\pi_n(i)}}}{\left(\sum_{j=i}^{K} p^n_{\pi_n(j)} + \tilde{\eta}^n_{\pi_n(i-1)}\right)^{w_{\pi_n(i-1)}}} \right) + w_{\pi_n(K)} \log_2 \left(\frac{1}{\tilde{\eta}^n_{\pi_n(K)}} \right) \Bigg].$$

Equality (b) comes from the definition (2). At (c), the weights $w_{\pi_n(i)}$ are put inside the logarithm. Finally, (d) is obtained by combining the numerator of the i-th term with the denominator of the $(i-1)$-th term, for $i \in \{2, \ldots, K\}$.

The constant term $w_{\pi_n(K)} \log_2 \left(1/\tilde{\eta}^n_{\pi_n(K)}\right)$ can be removed from the objective function, without loss of generality. By applying the change of variables (3), we derive the equivalent problem $\mathcal{P}'$. Constraints $C1'$ and $C2'$ are respectively equivalent to $C1$ and $C2$ since $x^n_1 = \sum_{j=1}^{K} p^n_{\pi_n(j)} = \sum_{k\in\mathcal{K}} p^n_k$, for $n \in \mathcal{N}$. Constraints $C3'$ and $C3''$ come from $C3$ and the fact that $x^n_i - x^n_{i+1} = p^n_{\pi_n(i)}$, for any $i \in \{1, \ldots, K\}$ and $n \in \mathcal{N}$. In the same way, the active users set in $C4'$ is defined as $\mathcal{U}'_n \triangleq \{i \in \{1, \ldots, K\} \colon x^n_i > x^n_{i+1}\}$. □

B. Proof of Theorem 3

Complexity analysis: The computational complexity mainly comes from the computation of V, X and U in lines 20 to 39. It requires $(M-1)\sum_{j=2}^{K} K+1-j = O(MK^2)$ iterations. Each iteration has a constant number of operations. Thus, the overall worst case computational complexity is $O(MK^2)$.

Optimality: The detailed analysis of Algorithm 2 in Section IV-A shows that SCUS's construction of V, X, U indeed match their definition. In particular, $V[M, K, K]$ is by definition the optimal value of $\mathcal{P}'_{SCUS}(n)$. The corresponding optimal allocation $\boldsymbol{x}^n$ is retrieved in lines 40 to 46. □

C. Proof of Theorem 4

Complexity analysis: At each for loop iteration i, the while loop at line 6 has at most i iterations. Thus, the worst case complexity is proportional to $\sum_{i=1}^{|\mathcal{U}'_n|} i = O(|\mathcal{U}'_n|^2)$.

Optimality: Without loss of generality, we can suppose that the $x^n_{i_n}$'s are initialized to zero. We will prove by induction that at the end of each iteration i at line 10, the following induction hypotheses are true:

$H_1(i)$: $\sum_{l=1}^{i} f^n_{l_n}$ is maximized by $x^n_{1_n}, \ldots, x^n_{i_n}$,
$H_2(i)$: Constraint $C3'$ is satisfied, i.e., $x^n_{1_n} \geq \cdots \geq x^n_{i_n} \geq 0$,
$H_3(i)$: For any $l \leq i$, there exists $q \leq l$ and $q' \geq l$ such that variables $x^n_{q_n}, \ldots, x^n_{q'_n}$ are equal and optimal for $f^n_{q_n,q'_n}$. In addition, $f^n_{q_n,q'_n}$ is decreasing for any $y^n_{q_n}, \ldots, y^n_{q'_n} \geq x^n_{q_n}, \ldots, x^n_{q'_n}$ such that $\bar{P}^n \geq x^n_{q_n} \geq \cdots \geq x^n_{q'_n}$.

We set $x_{K+1} = 0$, which satisfies $C3''$. $C2'$ is always satisfied since algorithm MAXF gives values in $\left[0, \bar{P}^n\right]$.

Basis: For $i = 1$, x^* computed at line 3 is indeed the optimal of $f^n_{1_n}$. The while loop has no effect since $j = 0 < 1$, therefore $x^n_{1_n} \leftarrow x^*$ and statements $H_1(i)$ and $H_2(i)$ are true. Since $f^n_{1_n}$ is decreasing for $x^n_{1_n} \in \left[x^*, \bar{P}^n\right]$ according to Lemma 2, $H_3(i)$ is also satisfied for $l = 1$ with $q = q' = 1$.

Inductive step: Assume that $x^n_{1_n}(i-1), \ldots, x^n_{(i-1)_n}(i-1)$ are the variables verifying $H_1(i-1)$ – $H_3(i-1)$ at iteration $i-1 < K$. Let the variables at iteration i be $x^n_{1_n}, \ldots, x^n_{i_n}$.

If by assigning $x^* = \text{MAXF}(i_n, i_n, \mathcal{I}^n, \bar{P}^n)$ to $x^n_{i_n}$, it satisfies $C3'$, then $\sum_{l=1}^{i-1} f^n_{l_n}(x^n_{l_n}(i-1)) + f^n_{i_n}(x^n_{i_n})$ is optimal due to $H_1(i-1)$ and the fact that $x^n_{i_n}$ is maximal for $f^n_{i_n}$. The iteration terminates with $x^n_{i_n} \leftarrow x^*$ and $x^n_{l_n} = x^n_{l_n}(i-1)$ for all $l < i$. Thus, $H_1(i)$ and $H_2(i)$ are satisfied and $H_3(i)$ is true for index $l = i$ with $q = q' = i$.

Otherwise if $x^n_{i_n} = x^*$ violates $C3'$, we have $x^n_{j_n}(i-1) < x^*$ at the while loop's first iteration at line 6 with $j \leftarrow i-1$. We deduce from the monotonicity of $f^n_{l_n}$, $l \leq i$, that:

$$x_{i_n} \geq x^n_{j_n}(i-1). \tag{7}$$

We know from $H_3(i-1)$ there exists $q \leq j$ such that $x^n_{q_n}(i-1), \ldots, x^n_{j_n}(i-1)$ are equal and optimal for $f^n_{q_n,j_n}$ and that $f^n_{q_n,j_n}$ is decreasing for any increase in these variables. It follows from Eqn. (7) that the optimal is reached at $x^n_{q_n} = \cdots = x^n_{j_n} = x_{i_n}$.

As a consequence, the maximum of $\sum_{l=q}^{i} f^n_{l_n} = f^n_{q_n,i_n}$ is given by $x^* \leftarrow \text{MAXF}(q_n, i_n, \mathcal{I}^n, \bar{P}^n)$. Then, the algorithm sets $x^n_{q_n}, \ldots, x^n_{i_n} \leftarrow x^*$ and $x^n_{l_n} = x^n_{l_n}(i-1)$ for all $l < q$. $\sum_{l=q}^{i} f^n_{l_n}(x^n_{l_n})$ is optimal by construction. $\sum_{l=1}^{q-1} f^n_{l_n}(x^n_{l_n})$ remains unchanged and optimal by induction hypothesis at iteration $q-1$. Therefore, $H_1(i)$ is true. Property $H_3(i)$ remains unchanged on $l < q$, and is also true for any $l \in \{q, \ldots, i\}$ with indexes q and $q' = i$. The only property that may not be satisfied at this point is $H_2(i)$, since $x^n_{(q-1)_n} < x^*$ is possible.

We continue this reasoning until reaching the highest index $j \in \{2, \ldots, i-1\}$ such that $x^n_{(j-1)_n} \geq x^* = \text{MAXF}(j_n, i_n, \mathcal{I}^n, \bar{P}^n)$. We know that this index exists since all variables are upper bounded by $\bar{P}^n$ and $x^n_{1_n} = \bar{P}^n$ due to Lemma 2. Properties $H_1(i)$ and $H_3(i)$ remain true by the above construction. Finally, $H_2(i)$ is satisfied at this point. □

D. Proof of Lemma 5

The convexity of $\mathcal{F}$ is direct from its definition in Eqn. (6). We now prove that each F^n is concave with respect to $\bar{P}^n \in [0, P^n_{max}]$, $n \in \mathcal{N}$. Let $x^n_{1_n}, \ldots, x^n_{|\mathcal{U}'_n|_n}$ be the allocation returned by SCPC$(\mathcal{I}^n, \mathcal{U}'_n, P^n_{max})$. We can see in Algorithm 3, that the only impact of $\bar{P}^n$ on the allocation is through MAXF thresholding (see lines 4 and 6 in Algorithm 1). Hence, the allocation returned by SCPC$(\mathcal{I}^n, \mathcal{U}'_n, \bar{P}^n)$ is equal to $\min\{x^n_{1_n}, \bar{P}^n\}, \ldots, \min\{x^n_{|\mathcal{U}'_n|_n}, \bar{P}^n\}$. Besides, the variables can be divided in consecutive groups of the form $x^n_{q_n}, \ldots, x^n_{q'_n}$, such that $x^n_{q_n} = \cdots = x^n_{q'_n}$ is optimal for $f^n_{q_n,q'_n}$, according to H_3 of Theorem 4's proof. Lemma 2 implies that $f^n_{q_n,q'_n}$ is concave on $\left[0, x^n_{q_n}\right]$, therefore $f^n_{q_n,q'_n}\left(\min\{x^n_{q_n}, \bar{P}^n\}\right)$ is concave in $\bar{P}^n \in [0, P^n_{max}]$. It follows by summation that F^n and $\sum_{n\in\mathcal{N}} F^n$ are concave, which completes the proof. □