SIR Estimation in Hexagonal Cellular Networks with Best Server Policy

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February 2, 2012

Abstract

The evaluation of the Signal to Interference Ratio (SIR) in cellular networks is of primary importance for network dimensioning. For static studies, which evaluate cell capacity and coverage, as well as for dynamic studies, which consider arrivals and departures of mobile stations (MS), the SIR is always an important input. Contrary to most of the analytical works evaluating SIR, we assume in this paper that the MS is attached to the *best server*, i.e., to the base station (BS) from which it receives the highest power. This is a more realistic policy compared to the classical one that considers MSs to be attached to the *nearest* BS. The exact formulation of the SIR is however in this case uneasy to handle and numerical methods remain heavy. In this paper, we thus propose an approximate analytical study on the average SIR and SIR distribution in lognormally shadowed networks based on truncated lognormal distributions that provides very close results with respect to Monte Carlo simulations.

1 Introduction

Downlink capacity estimation is an issue of critical importance in cellular networks design, and it is directly bonded with the Signal over Interference Ratio (SIR) estimation.

Several papers address the capacity estimation matter under the assumption that the Mobile Station (MS) is always served by the nearest Base Station (BS), although the presence of shadowing in real mobile networks configures a more complex scenario: in fact, according to the most commonly used policy the MS is served by the station from which it receives more power (*best server* policy), which can be often different than the nearest BS. The shadowing effect, affecting each signal received by the MS and usually modeled as a lognormal random variable [1], depends on the local environment in which the MS is deployed and cannot be ignored while trying to model real situations. Moreover, the performance indicators analyzed in the papers, are in some cases (e.g. TDMA/OFDMA systems) not ideal for the purpose of estimating the network capacity; for example, the Other-Cell Interference Factor (OCIF), defined as the ratio of the total in-cell received power to the out-of-cell received power is often derived in place of the SIR. OCIF is not a very useful parameter in CDMA systems, while in TDMA/OFDMA systems SIR is more meaningful.

In [2] and [3] the outage probability in presence of shadowing is derived without considering the *best server*, through the OCIF computation, while [4] and [5] compute the average OCIF (no *best server*) over a cell by numerical integration, considering a hexagonal network. In [6] the precise distribution of the other-cell interference is derived, although the derived formulas are quite difficult to be implemented.

The probability density function (PDF) and the cumulative density function (CDF) of the SIR in presence of fast fading and shadowing is found in [7], under the assumption that all the interferers have the same mean, which is unrealistic in real networks, where each BS has a different distance with respect to the MS. This assumption is not used in [8], but the formulae derived for the SIR distribution require the knowledge of the serving station, which is not given when the *best server* policy is used, and which depends on the channel conditions. [9] and [10] find the OCIF (no *best server* taken into account) for CDMA systems, using simulation and modeling.

In this paper, we extend the results presented in [11], where both the shadowing effect and the best server policy were taken into account to derive the average SIR, by introducing an approximate method to derive the SIR distribution and new results and analysis concerning the average SIR estimation. Section 2 introduces the system model, while in Section 3 the SIR expression is derived, making use of the probability for an MS to be served by a given station in the network, whose expression is shown in a dedicated subsection. An approximate method for the derivation of the average SIR, making use of the fluid network model [12] is then exposed in Section 4 and validated in Section 6 through Monte-Carlo simulations, for a wide set of path-loss models and shadowing standard deviations. Section 5 illustrates an approximate method for the derivation of the SIR distribution, based on truncated lognormals; the validation of this method is accomplished in Section 6.

2 System Model

We consider the downlink of a hexagonal radio cellular network. All the BSs have omni-directional transmitting antennas and they transmit using the same power P_{tx}, while a frequency reuse 1 pattern is adopted, meaning that all the stations transmit on the same frequency. The BSs density is ρ_{BS}, the half-distance between BSs is R_c and the cell range is R.

All the MSs are assumed to be served by the BS from which they receive more power, measured on the pilot signal, according to the so-called *best server* policy. MS location in the network can be measured by taking the serving BS (position in the logical cell) or the nearest BS (position in the geographical cell) as origin. In this paper we always refer to the geographical cell because it represents a steady reference, while the logical cell changes according to the serving station, which in turn depends on the channel conditions.

The metric for the evaluation of the transmission quality is the average SIR $\overline{\gamma}(d)$ experienced by an MS, expressed as a function of the distance d from the nearest BS.

The propagation of radio signals is supposed to be affected by path-loss and lognormal shadowing, according to the following model:

$$S_i(d) = P_{tx} K {r_i}^{-\eta} A, \tag{1}$$

where $S_i(d)$ represents the received power from the i-th BS by an MS at a distance d from the cell center, K is a constant, r_i is the distance between the considered MS and the i-th BS, η is the path loss exponent and $A = 10^{\frac{\xi}{10}}$ is a lognormal random variable (RV) taking into account the variations over the received power due to the shadowing effect. ξ is a normal zero-mean RV, whose standard deviation is denoted with σ (in dB). Received power S_i is thus a lognormal RV with PDF f_{S_i} and CDF Φ_i. The background noise is not considered here, because its effects are neglectable with respect to the interference effect in urban environments.

The SIR is evaluated according to the position of a given MS in its cell, conventionally considered to be the 'central cell' of the network and covered by a BS conventionally named as 'BS1' (figure 1).

Let us divide the central cell into 6 sectors, each one covering an angle of 60°, having its vertex in the BS1, as shown in figure 1. The borders of each sector are the lines joining the BS1 with its nearest neighbors BSs. Without any loss of generality, the generic MS we are considering in this paper can be assumed to be deployed in one of these sectors, thanks to the symmetries of the hexagonal network. As a further simplification, we assume MS to lie on the line splitting the chosen sector into two symmetric sectors of 30° each (figure 1). This assumption doesn't meaningfully affect the validity of the obtained results, as shown in Section 6, and thanks to it we can exploit many symmetries to derive the average received powers from each BS.

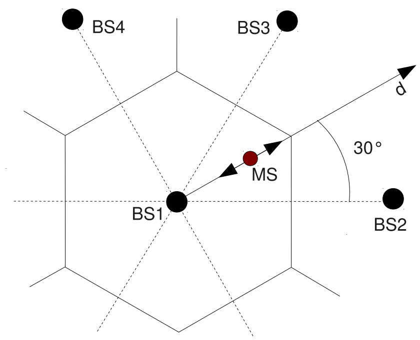

Figure 1: System model

3 Average SIR

This section is dedicated to the computation of the average SIR experienced by an MS at a given distance d from the cell center. For the sake of notation simplicity the index d is omitted in the following.

In presence of shadowing, an MS could be served by any of the BSs. This makes the SIR computation quite complex in absence of further assumptions, because the serving BS choice strongly influences the experienced SIR. Let now suppose we know the probability p_i for a given MS to be served by the i-th BS. We can compute the average SIR $\overline{\gamma}_i$ knowing that the serving BS is BS i and then $\overline{\gamma}$ can be derived summing all the $\overline{\gamma}_i$ for every possible serving station i, weighting each of them with its associated p_i:

$$\overline{\gamma} = \sum_{i \in \mathcal{B}} \overline{\gamma}_i p_i, \tag{2}$$

where $\mathcal{B}$ of cardinality B is a set including all the indexes of the network BSs.

In Subsections 3.1 and 3.2 the expressions for p_i and $\overline{\gamma}_i$ are derived.

3.1 Derivation of p_i

The terms $p_i, i \in \mathcal{B}$ are key quantities for the derivation of $\overline{\gamma}$, because they represent the 'weights' to be given to each term $\overline{\gamma}_i$. Their computation is based on statistical considerations.

We consider the quantity L_k, defined as the power S_k received from the k-th BS expressed in dBm:

$$L_k = 10 \log_{10}(S_k). \tag{3}$$

Given the lognormal distribution of S_k, L_k is a normal-distributed RV, whose PDF f_{L_k} is given by

$$f_{L_k}(x) = \frac{1}{\sqrt{2\pi a^2\sigma^2}} exp\left(-\frac{(x-\mu_k)^2}{2a^2\sigma^2}\right), \qquad (4)$$

where $\mu_k = 10\log_{10}(P_{tx}Kr_k^{-\eta})$ is the average value of L_k, and $a = \ln 10/10$.

The probability p_i for an MS to be served by the i-th BS can be written as

$$p_i = P(S_i > S_h,\ \forall h \neq i) = P(L_i > L_h,\ \forall h \neq i), \qquad (5)$$

which gives, taking into account the independence between all the L_k,

$$\begin{aligned} p_i &= \int_{-\infty}^{+\infty} P(L_h < x,\ \forall h \neq i) f_{L_i}(x)dx \\ &= \int_{-\infty}^{+\infty} \prod_{h\neq i} Q\left(\frac{x-\mu_h}{a\sigma}\right) f_{L_i}(x)dx, \end{aligned} \qquad (6)$$

where $Q(x) = 1/\sqrt{2\pi}\int_u^{+\infty} e^{-u^2/2}du$ is the error function. Note that the computation of p_i is very fast because few samples of the normal distribution are needed in order to have an accurate approximation of the integral, e.g. with the rectangle rule [13].

3.2 Derivation of $\overline{\gamma}_i$

Parameter $\overline{\gamma}_i, i \in \mathcal{B}$ denotes the SIR when P_i is the highest received power. To derive it, we introduce the random variables $\tilde{S}_{h,i}$, which represent the received powers from BSs $h \neq i$ knowing that BS i is the serving BS:

$$\tilde{S}_{h,i} = \frac{S_h \mathbf{1}_{\{S_z < S_i, \forall z \neq i\}}}{p_i}. \qquad (7)$$

We also define the interference power knowing that BS i is the serving BS:

$$\tilde{I}_i = \sum_{h\neq i} \tilde{S}_{h,i}. \qquad (8)$$

With these notations, $\overline{\gamma}_i$ can be written:

$$\overline{\gamma}_i = E\left[\frac{\tilde{S}_{i,i}}{\tilde{I}_i}\right] \qquad (9)$$

$$= \int_0^{+\infty} xE\left[\frac{1}{\tilde{I}_i}\,\middle|\,\tilde{S}_{i,i} = x\right] f_{\tilde{S}_{i,i}}(x)dx, \qquad (10)$$

where $f_{\tilde{S}_{i,i}}(x)$ is the PDF of the power received by the serving station.

On the one hand, this function can be obtained considering that

$$\begin{aligned} f_{\tilde{S}_{i,i}}(x)dx &= P(S_i = x \,|\, S_i > S_h, \forall h \neq i)dx \\ &= P\left(S_i > S_h, \forall h \neq i \,|\, S_i = x\right)\frac{P(S_i = x)dx}{p_i} \\ &= \prod_{h\neq i} \Phi_h(x)\frac{f_{s_i}(x)}{p_i}dx, \end{aligned} \qquad (11)$$

where we exploited the fact that RVs S_h are independent.

On the other hand, the expectation in (10) can be written as

$$E\left[\frac{1}{\tilde{I}_i}\middle|\tilde{S}_{i,i}=x\right]=\int\frac{1}{t}f_{\tilde{I}_i\,|\,\tilde{S}_{i,i}=x}dt. \tag{12}$$

The function $f_{\tilde{I}_i|\tilde{S}_{i,i}=x}$ is the PDF of a sum of $B-1$ lognormal RVs truncated in x:

$$f_{\tilde{I}_i|\tilde{S}_{i,i}=x}=f_{\hat{S}_{1,x}}\ast\cdots\ast f_{\hat{S}_{i-1,x}}\ast f_{\hat{S}_{i+1,x}}\ast\cdots\ast f_{\hat{S}_{B,x}}, \tag{13}$$

where $\ast$ is the convolution operator and

$$f_{\hat{S}_{h,x}}(y)=\frac{f_{S_h}(y)}{P(S_h<x)},\ 0<y<x,\ h\neq i. \tag{14}$$

We finally rewrite (10), taking into account (11), as

$$\overline{\gamma}_i = \iint_0^{+\infty}\frac{x}{t}f_{\tilde{S}_{i,i}}(x)f_{\tilde{I}_i|\tilde{S}_{i,i}=x}(t)dxdt \tag{15}$$

$$= \iint_0^{+\infty}\frac{x}{t}\mathop{\ast}_{h\neq i}f_{S_h}(t)\frac{f_{S_i}(x)}{p_i}dxdt. \tag{16}$$

We now observe that p_i can be simplified in equation (2):

$$\overline{\gamma}=\sum_{i\in\mathcal{B}}\iint_0^{+\infty}\frac{x}{t}\mathop{\ast}_{h\neq i}f_{S_h}(t)f_{S_i}(x)dxdt. \tag{17}$$

It is now theoretically possible to compute the average SIR using this formula. However, the convolution over the whole set of BSs can be difficult to be performed in a large network. In the same way, infinite integrals computations are not easily performed for practical shadowing standard deviations because of the support of a lognormal PDF. Several approximations are however possible and give valid results, as shown in the next section.

4 Approximation Approach for $\overline{\gamma}$

We derive here an approximate method, valid on a wide range of values of σ and η, which makes the computation of $\overline{\gamma}_i$ sensibly simpler.

Let divide the set $\mathcal{B}$ in 2 sets, $\mathcal{B}=\mathcal{B}_n\bigcup\mathcal{B}_f$, where $\mathcal{B}_n$ of cardinality N is the set of the indexes of the N nearest BSs, with respect to the central cell, while $\mathcal{B}_f$ of cardinality $B-N$ includes the indexes of all the other BSs in the network. According to (2), $\overline{\gamma}_i$ is obtained as the sum of B terms, where B is the number of BSs in the network and is thus potentially infinite. However, for all the practical values of η, the probabilities p_i are meaningful only for the nearest BSs, whose indexes are included in $\mathcal{B}_n$. Thus we can write that

$$\overline{\gamma}\approx\sum_{i\in\mathcal{B}_n}\overline{\gamma}_ip_i. \tag{18}$$

4.1 Approximation of $E\left[\tilde{S}_{i,i}/\tilde{I}_i\right]$

The rigorous computation of $\overline{\gamma}_i = E\left[\tilde{S}_{i,i}/\tilde{I}_i\right]$ is not immediate and includes integrals and convolutions. However, several simplifications are valid in most of the practical cases, where some reasonable assumptions can be made. The approximations introduced in the following hold when σ is low and p_i is high, and their accuracy improves for decreasing σ and increasing p_i.

4.1.1 Approximate independence of $\tilde{S}_{i,i}$ and $\tilde{I}_i$

Let us consider the PDFs of $\tilde{S}_{i,i}$ and $\tilde{I}_i$. Each power S_i received by the i-th BS is independent from the powers received by the other BSs, but if we assume to know the identity of the serving station this independence is lost: all the values of the interferers received powers S_h are forced to be lower than the serving station received power S_i. However, when p_i is high and σ is small, the $\tilde{S}_{i,i}$ PDF support is separated or almost separated with respect to each $\tilde{S}_{h,i}, h \neq i$ PDF support. Hence, the probability that a given realization of $\tilde{S}_{i,i}$ will force the PDF of any $\tilde{S}_{h,i}, h \neq i$ to meaningfully change is low. For this reason we approximate $\tilde{S}_{i,i}$ and $\tilde{I}_i$ as two independent RVs:

$$E\left[\frac{\tilde{S}_{i,i}}{\tilde{I}_i}\right] \approx E\left[\tilde{S}_{i,i}\right] E\left[\frac{1}{\tilde{I}_i}\right]. \tag{19}$$

Figure (2) depicts the PDFs of $\tilde{S}_{1,1}$ and $\tilde{S}_{h,1}$, where the h-th BS is the nearest neighbor BS (the one from which the MS receives the highest average power, after the central BS), for $\sigma = 3$ dB. We notice that even the strongest interferer PDF is almost 'separated' with respect to the serving station one, when σ is low and p_i is high, letting us consider $\tilde{S}_{i,i}$ and each $\tilde{S}_{h,i}$ as approximately independent.

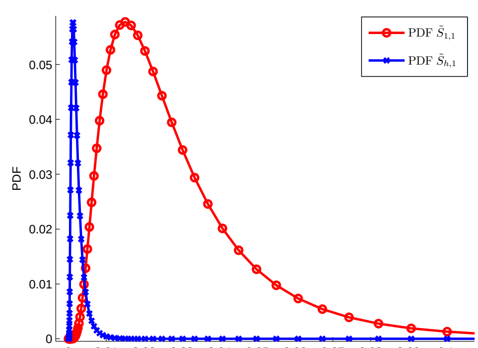

Figure 2: PDFs of $\tilde{S}_{1,1}$ and $\tilde{S}_{h,1}$ ($\eta = 2.5$, $\sigma = 3$ dB and $d = 475$ m).

4.1.2 Approximation of $E\left[\frac{1}{\tilde{I}_i}\right]$ with $\frac{1}{E\left[\tilde{I}_i\right]}$.

Applying the causal form of the central limit theorem [14] we can approximate the distribution of $\tilde{I}_i$, which is the sum of B_n positive random variables, with a gamma distribution:

$$f_{\tilde{I}_i}(x) \approx \frac{x^{v-1}}{\Gamma(v)\lambda^v} e^{-\frac{x}{\lambda}}, \tag{20}$$

where $v = E\left[\tilde{I}_i\right]^2 / \mathrm{var}(\tilde{I}_i)$ is the shape parameter, while $\Gamma(*)$ represents the gamma function [14] and λ is the scale parameter. Analysing the term v we notice that for decreasing σ its value tends to increase, as demonstrated in Figure (3).

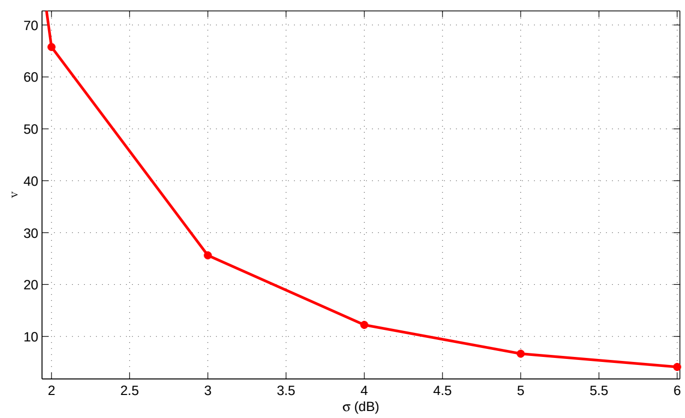

Figure 3: Value of the parameter v vs σ, obtained using Monte-Carlo simulations ($\eta = 2.5$ and $d = 325$ m).

When v is large the gamma distribution can be well approximated with a normal distribution [14]. Hence, under the conditions we have supposed, the term $\tilde{I}_i$ is close in distribution to a normal RV.

Let us now analyze the ratio $1/\tilde{I}_i$. Applying Jensen's inequality [15] we obtain a useful bound on its expectation:

$$E\left[\frac{1}{\tilde{I}_i}\right] \geqslant \frac{1}{E\left[\tilde{I}_i\right]}. \tag{21}$$

When the conditions for the approximate normality of $\tilde{I}_i$ hold, it is possible to apply the delta method [16] to get an esteem of $E\left[1/\tilde{I}_i\right]$. The delta method states that if a RV X_n satisfies

$$\sqrt{n}\left[X_n - \theta\right] \xrightarrow{D} \mathcal{N}\left(0, \sigma^2\right), \tag{22}$$

where θ and σ^2 are finite value constants and $\xrightarrow{D}$ denotes convergence in distribution, it is the case that

$$\sqrt{n}\left[g\left(X_n\right) - g\left(\theta\right)\right] \xrightarrow{D} \mathcal{N}\left(0, \sigma^2\left[g\prime\left(\theta\right)\right]^2\right). \tag{23}$$

Thus, for decreasing σ the value of $E\left[\frac{1}{\tilde{I}_i}\right]$ will get closer to the bound given by (21), as shown in Figure 4, where a comparison between $E\left[\frac{1}{\tilde{I}_i}\right]$ and $\frac{1}{E[\tilde{I}_i]}$ is drawn, considering an MS served by the central BS (high p_i). We can notice how, for low σ, $E\left[\frac{1}{\tilde{I}_i}\right]$ can be well approximated by its lower bound:

$$E\left[\frac{1}{\tilde{I}_i}\right] \approx \frac{1}{E\left[\tilde{I}_i\right]}. \tag{24}$$

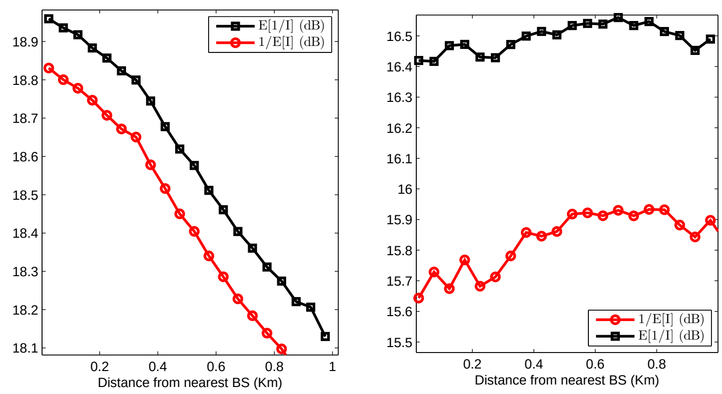

Figure 4: $E\left[\frac{1}{\tilde{I}_i}\right]$ vs $\frac{1}{E[\tilde{I}_i]}$ for an MS served by the central BS ($\eta = 2.5$, $\sigma = 3$ dB (left) and $\sigma = 6$ dB (right)).

4.2 Computation of $E\left[\tilde{S}_{i,i}\right]$ and $E\left[\tilde{I}_i\right]$

According to the discussion developed in 4.1.1 and 4.1.2 we can finally state that under the mentioned conditions the following approximation holds:

$$E\left[\frac{\tilde{S}_{i,i}}{\tilde{I}_i}\right] \approx \frac{E\left[\tilde{S}_{i,i}\right]}{E\left[\tilde{I}_i\right]} \tag{25}$$

We now focus on each of these expectations by approximating involved RVs by truncated lognormal RVs.

4.2.1 Average Received Signal

We know that $\tilde{S}_{i,i}$ can be derived as:

$$E\left[\tilde{S}_{i,i}\right] = \frac{1}{p_i}\int_0^{+\infty} x f_{S_i}(x) \prod_{h \neq i} \Phi_h(x) dx. \tag{26}$$

Let us consider the term $\omega(x) = \prod_{h \neq i} \Phi_h(x)$ in Equation (26), which is strictly increasing by definition, and whose value is zero for $x = 0$ and 1 for $x \to +\infty$. If p_i is high we can reasonably suppose $\omega(x)$ to be close to 1 for most of the support of S_i, having a sharp decrease while getting close to zero. An effective approximation, validated through simulations, is to substitute $\omega(x)$ with a heaviside step function:

$$\omega(x) = \prod_{h \neq i} \Phi_h(x) \approx \begin{cases} 0 & \text{for} \quad x < \overline{r}_i \\ 1 & \text{for} \quad x \geqslant \overline{r}_i \end{cases} \tag{27}$$

Here $\overline{r}_i$, representing the location of the step, has been chosen in order to maintain the value of the integral $\int_0^{+\infty} \prod_{h \neq i} \Phi_h(x) f_{S_i}(x) dx$ equal to p_i, i.e.

$$\overline{r}_i = \Phi_i^{-1}(1 - p_i). \tag{28}$$

Hence, $E\left[\tilde{S}_{i,i}\right]$ can be approximated as

$$E\left[\tilde{S}_{i,i}\right] \approx \frac{1}{p_i} \int_{\overline{r}_i}^{+\infty} x f_{S_i}(x) dx \tag{29}$$

$$= E\left[S_i \mid S_i > \overline{r}_i\right]. \tag{30}$$

From Eq. (30) it follows that the PDF of $\tilde{S}_{i,i}$ is approximated with the pdf of S_i truncated in $\overline{r}$, which is equivalent to suppose zero probability for the event $\{\tilde{S}_{i,i} < \overline{r}_i\}$. The computation of $E\left[\tilde{S}_{i,i}\right]$ exploiting Eq. (30) is reduced to the computation of the average of a lower truncated lognormal [17]:

$$E\left[\tilde{S}_{i,i}\right] \approx exp(\mu_i + \sigma^2/2) \frac{1 - \Phi_i\left(\frac{\mu_i + \sigma^2 - \ln(\overline{r}_i)}{\sigma}\right)}{1 - \Phi_i\left(\frac{\mu_i - \ln(\overline{r})}{\sigma}\right)}. \tag{31}$$

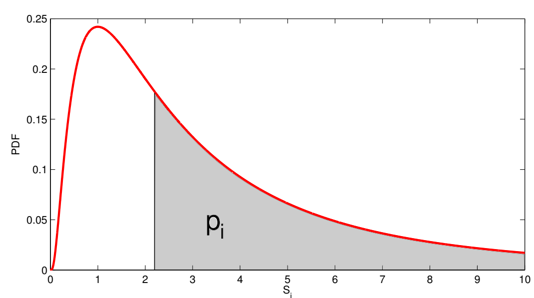

Figure 5: Truncated lognormal.

4.2.2 Average Interference

Let us now consider the average interference. $E[\tilde{I}_i]$ takes into account the average interference seen by an MS served by the i-th station. We compute it as the sum of two terms, one taking into account the contribution of the *far* BSs (included in $\mathcal{B}_f$) and one taking into account the contribution of the *near* BSs (included in $\mathcal{B}_n$). Only the PDF of the power received by the BSs belonging to $\mathcal{B}_n$

are considered to be meaningfully modified with respect to the lognormal distribution, when the serving station is known. Thus the contribution I_f of the *far* BSs can be approximated with the average of a sum of lognormal RV:

$$E[\tilde{I}_i] \approx E\left[\sum_{h\in\mathcal{B}_n} \tilde{S}_{h,i}\right] + E\left[\sum_{h\in\mathcal{B}_f} S_h\right] \tag{32}$$

$$= \sum_{h\in\mathcal{B}_n} E\left[S_h \middle|\ S_i > S_h, \forall h \neq i\right] + I_f. \tag{33}$$

The precision of the approximation is determined by N. The term $I_f = \sum_{h\in\mathcal{B}_f} E\left[S_h\right]$ can be computed starting from the expression found in [18], which well approximates the extra-cell received power without shadowing nor *best server* policy, using the fluid model. Following this approach, taking into account the shadowing effect and removing the influence of the nearest BSs, we obtain:

$$\begin{aligned} I_f &= \int_{2R_c-d}^{+\infty}\int_0^{2\pi} \rho_{BS} P_{tx} K r^{-\eta} A r dr d\theta - E\left[\sum_{h\in\mathcal{B}_n} S_h\right] \\ &= \frac{2\pi\rho_{BS}P_{tx}K}{\eta-2}\left[(2R_c-d)^{2-\eta}\right] e^{\frac{a^2\sigma^2}{2}} \\ &- e^{\frac{a^2\sigma^2}{2}} \sum_{h\in\mathcal{B}_n} e^{\mu_h} \end{aligned} \tag{34}$$

Concerning the first term of the sum (32), we adopt the same approach as for the average received power. Each of the terms $E\left[\tilde{S}_{h,i}\right]$ can be written as:

$$E\left[\tilde{S}_{h,i}\right] = \int_0^{+\infty} x f_{S_h}(x) \frac{P(\hat{S}_{h,i} > x)}{P(\hat{S}_{h,i} > S_h)} dx, \tag{35}$$

where

$$P\left(\hat{S}_{h,i} = x\right) = P\left(S_h = x \mid S_h < S_i \cap S_i > S_z, \forall z \neq \{i,h\}\right). \tag{36}$$

For increasing p_i the RV $\hat{S}_{h,i}$ becomes closer to $\tilde{S}_{h,i}$, so for the terms p_i associated to the BSs belonging to the set $\mathcal{B}_n$ we introduce the approximation:

$$E\left[\tilde{S}_{h,i}\right] \approx \int_0^{+\infty} x f_{S_h}(x) \frac{P(\tilde{S}_{i,i} > x)}{P(\tilde{S}_{i,i} > S_h)} dx. \tag{37}$$

Now, let consider the term $\omega'(x) = P(\tilde{S}_{i,i} > x)$. Following the same approach of the previous section, we aim to find a heaviside step function approximation for $\omega'(x)$, in order to simplify the computations. The criterion we adopt to set the new step value $\overline{r}'_i$ is to minimize the square of the difference between the approximating function and the original function i.e. choosing as step point the value where we have the median value of the CDF of $\tilde{S}_{i,i}$. Thus $\overline{r}'_i$ is obtained as

$$\overline{r}'_i = \Phi_i^{-1}(1 - p_i/2), \tag{38}$$

and the term $E\left[\tilde{S}_{h,i}\right]$ is finally replaced by

$$E\left[\tilde{S}_{h,i}\right] \approx \frac{1}{P(\tilde{S}_{i,i} > S_h)} \int_0^{\overline{r}'_i} x f_{S_h}(x) dx \tag{39}$$

$$= E\left[S_h \,\middle|\, S_h < \overline{r}'_i\right], \tag{40}$$

whose computation is straightforward, being the average of an upper truncated lognormal:

$$E\left[\tilde{S}_{h,i}\right] \approx exp(\mu_h + a^2\sigma^2/2) \frac{\Phi_h\left(\frac{\mu_h + \sigma^2 - \ln(\overline{r}'_i)}{a\sigma}\right)}{\Phi_h\left(\frac{\mu_h - \ln(\overline{r}'_i)}{a\sigma}\right)}. \tag{41}$$

Simulations in Section 6 show that this value provides a very good approximation.

Summarizing, the approximate method requires, for every average SIR value we want to estimate, the calculation of few formulas to evaluate the contribution of each of the N terms considered in the derivation of $\overline{\gamma}$. These operations include the computation of the average signal (according to formula (31)), the computation of I_f (applying formula (34)), and the computation of the N approximate values of $E\left[\tilde{S}_{h,i}\right]$ (each derived as in formula (41)). All the computationally unprofitable integrals over the support of a lognormal RV are in this way avoided.

5 SIR Distribution

The aim of this section is to find a good approximation for the SIR average CDF at a given distance from the cell center. Let us introduce the RV γ, representing the SIR ratio. We remind that $E[\gamma] = \overline{\gamma}$ By analogy with (2) the SIR CDF Φ_γ can be written as:

$$\Phi_\gamma(x) = P(\gamma < x) = \sum_{i \in \mathcal{B}} p_i \Phi_{\gamma_i}(x), \tag{42}$$

where $\Phi_{\gamma_i}(x)$ is the SIR CDF knowing the serving station:

$$\Phi_{\gamma_i}(x) = P\left(\frac{S_i}{\sum_{h \neq i} S_h} < x \,\middle|\, S_i > S_h, \forall h \neq i\right). \tag{43}$$

The analytical derivation of Φ_{γ_i} is not immediate and is out of the scope of this paper, while an approximate method is proposed.

We define $\breve{\gamma}_i$ as the ratio of S_i with respect to the sum of all the other received powers, without any further knowledge:

$$\breve{\gamma}_i = \frac{S_i}{\sum_{j \neq i} S_j}. \tag{44}$$

Now, following the same reasoning previously developed for the approximate $\tilde{S}_{i,i}$ distribution and $\tilde{S}_{h,i}$ distribution, we suppose that the probability for an MS to be served by a BS providing a low SIR is very small and can be neglected. Thus, the distribution of γ_i can be approximated with the distribution of $\breve{\gamma}_i$ lower truncated. This is equivalent to suppose that all the MSs which experienced SIR with respect to BS i is lower than a certain threshold $\overline{r}_{\gamma_i}$ will be served by another BS.

Parameter $\breve{\gamma}_i$ is the ratio of a lognormal RV and the sum of several lognormal RVs. According to Fenton-Wilkinson [19] a sum of lognormal RVs can be effectively approximated with another lognormal RV, if σ is low, letting us express $\breve{\gamma}_i$ as the ratio of two lognormal RVs, which gives in turn another lognormal RV. Thus, we can approximate γ_i as a lognormal RV lower truncated in $\overline{r}_{\gamma_i}$. Given that the value of σ is the same for all the lognormal RVs involved in the computation of $\breve{\gamma}_i$, and exploiting the Fenton-Wilkinson approximation [19], the parameters $\mu_{\breve{\gamma}_i}$ and $\sigma^2_{\breve{\gamma}_i}$ can be obtained as

$$\sigma^2_{\breve{\gamma}_i} = a^2\sigma^2 + \ln\left[\left(e^{a^2\sigma^2} - 1\right)\frac{\sum_{h\neq i} e^{2\mu_h}}{\left(\sum_{h\neq i} e^{\mu_h}\right)^2} + 1\right]; \tag{45}$$

$$\mu_{\breve{\gamma}_i} = \mu_i - \ln\left[\sum_{h\neq i} e^{\mu_h}\right] - \frac{a^2\sigma^2}{2} + \ln\left[\left(e^{a^2\sigma^2} - 1\right)\frac{\sum_{h\neq i} e^{2\mu_h}}{\left(\sum_{h\neq i} e^{\mu_h}\right)^2} + 1\right]. \tag{46}$$

We propose the following formula to establish the truncation point:

$$\overline{r}_{\gamma_i} = \Phi_i^{-1}(k(1-p_i)), \tag{47}$$

where $0 \leqslant k \leqslant 1$ is a coefficient to be tuned in order to find the best value of $\overline{r}_{\gamma_i}$.

The proposed approximated method for the SIR CDF shows good accuracy compared to Monte-Carlo simulations for low σ, as shown in Section 6, while for high σ there is less precision because of the higher interference variance. In this case the assumption that all the MSs with low SIR are served by another BS is far from the reality: in fact, it often happens that an MS receiving a good signal power from its serving BS is also experiencing a very high total interference power, giving in turn a low SIR. Hence the SIR PDF for high σ is 'smoother' than a truncated lognormal.

Also, the Fenton-Wilkinson approximation method accuracy decreases when σ grows, causing errors in the evaluation of the PDF of $\bar{\gamma}_i$.

6 Numerical Results

In this section, we compare our method to Monte Carlo simulations, and analyze the parameters influencing the results accuracy.

6.1 Simulation Model

All Monte-Carlo simulations are carried out considering a four rings hexagonal network (with cell range $R = 1$ Km) with wraparound. At each snapshot, an MS is dropped randomly with uniform spatial distribution in each sector. SIR is computed and averaged at a distance d from the BS over a ring of ranges d and $d + \gamma$ ($\gamma = 50$ m). Note that distance d is the distance between MS and the BS of its geographical cell (nearest BS) although SIR is computed with the *best server* policy. Simulations are performed with 10000 snapshots.

The path loss has been computed starting from the urban and suburban areas path loss model described in [20], where we assumed a BS height about the average rooftop $h_{BS} = 32m$ and a carrier frequency $f_c = 2.5GHz$, and we let the parameter η vary (in contrast to the original model where η is equal to 3.488) obtaining:

$$PL(r)|_{[dB]} = 10\eta \log_{10}(r) + 124, \tag{48}$$

where PL denotes the path loss, expressed in dB, and r is the distance between the MS and the considered BS. The use of this model is the intuitive reason for the hyperbolic-like shaped curves obtained in the Monte-Carlo simulations.

6.2 Simulation vs. Analysis: Average SIR

Contrary to the Monte-Carlo simulations, the analytical method considers only MSs lying on a line, as explained in Section 2, and thus it benefits from the symmetries given by the position of the MS in the proposed system model. In fact, the average received powers μ_i have to be computed only for half of the N near BSs, because each BS i different than the central one has a symmetric BS j with respect to the MS, whose BS-MS distance is the same, i.e. $r_i = r_j$, which implies $P_i = P_j$. Each term r_i is then derived by mean of basic geometric considerations. Figures (6) and (7) show the comparison between the analytical approximation and the simulation results for different figures of the path-loss exponent η and the shadowing standard deviation σ. In all cases, the matching is very good, even for $\sigma = 10$ dB. It is interesting to see that the average SIR is increasing with σ when the *best server* policy is chosen. In this case, the shadowing effect gives the MS an opportunity to receive a strong signal power from at least one of the neighboring BSs.

6.3 Simulation vs. Analysis: SIR Distribution

Figure (8) shows the comparison between the SIR CDF obtained from Monte-Carlo simulations and the one obtained from the approximate method of Section 5, for $\eta = 2.5$, $k = 0.9$, $d = 825$ m in the left figure and $d = 475$ m in the right figure. We notice how the accuracy if high, even for very challenging conditions (e.g. in the left figure we are near the cell border, η is low and one of the curves is drawn for a quite high value of σ).

6.4 Approximation Accuracy

The SIR derivation and approximate method for $\overline{\gamma}$ proposed in this paper are based on the assumption that the MS lies in a specific position for each given distance d (as described in Section 2), instead of considering the average of the SIR computed over a circle of ray d. In Figure (9) we

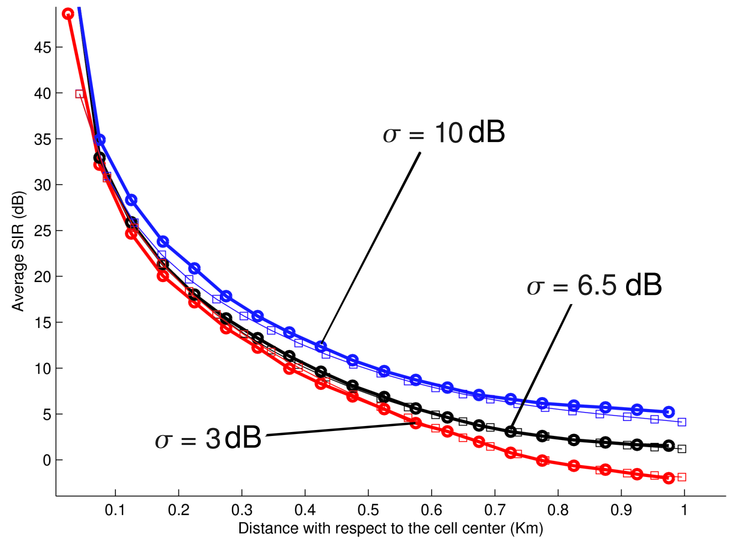

Figure 6: Average SIR using the *best server* policy vs. distance to the geographical cell center, approximate analysis (solid lines with circles) vs. simulations (solid lines with squares) ($\eta = 3$, $N = 18$).

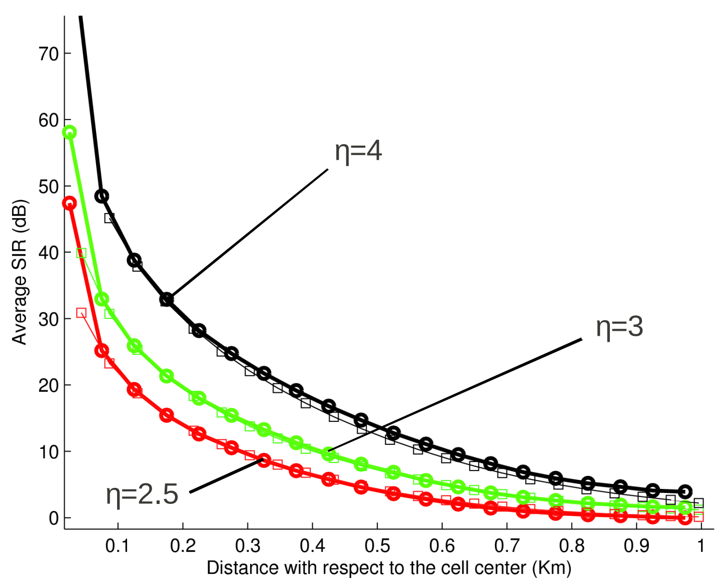

Figure 7: Average SIR using the *best server* policy vs. distance to the geographical cell center, approximate analysis (solid lines with circles) vs. simulations (solid lines with squares) ($\sigma = 6.5$ dB, $N = 18$).

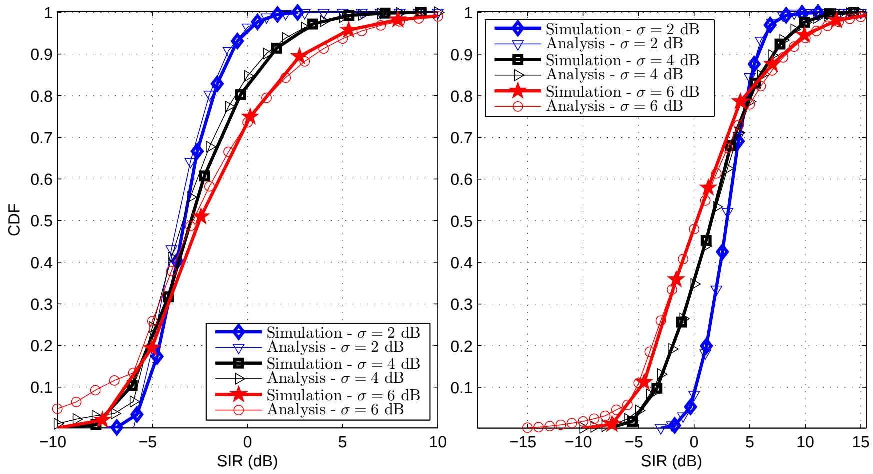

Figure 8: CDF of the SIR, approximate method vs. simulations ($\eta = 2.5$, $k = 0.9$). $d = 825$ m for the left figure, and $d = 475$ m for the right figure.

have plotted the average SIR as a function of the distance d from the center, for the cases of MSs unformly distributed over a circle of ray d around the cell center and MSs lying on the line specified in the model description. Results show that the assumption made does not produce important errors.

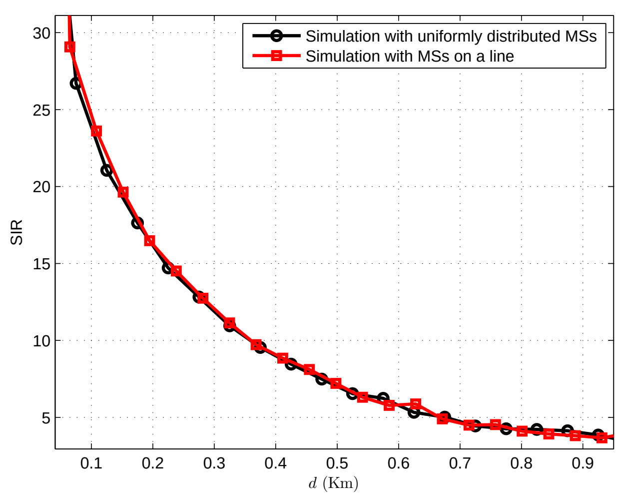

Figure 9: SIR vs. distance to cell center.

Figure (10) shows the accuracy of the approximations done for the computation of the average received signal (right) and of the average interference (left). On the y axis, we plot the difference between the simulated value and the approximation obtained with the truncated lognormal:

$10\log_{10}\left|E\left[\tilde{S}_i\right] - E\left[S_i \mid S_i > \bar{r}_i\right]\right|$ as a function of the cut off value $\bar{r}_i$ in logarithmic scale. The vertical line corresponds to $\bar{r}_i = \Phi_i^{-1}(1-p_i)$. In the same way, we plot $10\log_{10}\left|\sum\left(E\left[\tilde{S}_{h,i}\right] - E\left[S_h \mid S_h < \bar{r}'_i\right]\right)\right.$ as a function of $\bar{r}'_i$. The vertical line corresponds to $\bar{r}'_i = 1 - \Phi_i^{-1}(1-p_i/2)$. In both cases, the induced error is very low and our cut off values are nearly optimal. Errors have been plotted for specific figures of η, σ and d but the same conclusion can be drawn for a wide range of realistic values of these parameters.

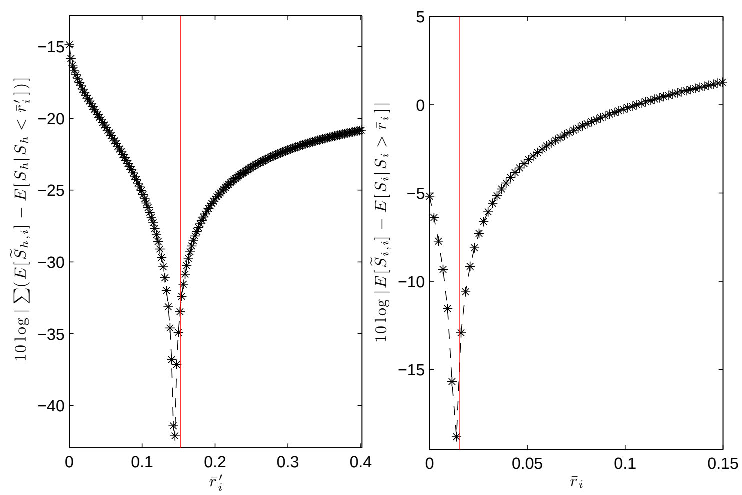

Figure 10: Truncated lognormal approximation error vs. cut off value for average received power (right) and interference (left) ($\eta = 4$, $\sigma = 10$ dB, $d = 0.5$ Km).

6.5 N Parameter Analysis

The cardinality of $\mathcal{B}_n$, indicated with N, is of fundamental importance for the results accuracy, and it is equal to the number of BSs for which the corresponding p_i is high enough to make the contribution of the associated $\overline{\gamma}_i$ meaningful. In Figure (11) we show how N can be kept very small for low values of σ, as only the 3 nearest BSs can reasonably become servers. For high σ the situation becomes more complex, and we need to take into account many stations. The value of η also plays an important role, determining the proportion between the powers received by the various BSs (the smaller η, the more comparable are the powers received).

Figure (12) shows the improvement of the results accuracy while increasing N. The value of N can be increased or decreased according to the desired trade-off between computational speed and precision of the results. The higher is σ, the higher should be N for a given accuracy because the high variations of the received power increase the influence of farther BSs. For low sigmas (e.g. $\sigma \approx 3$ dB) N can be set equal to 2 keeping a good results accuracy.

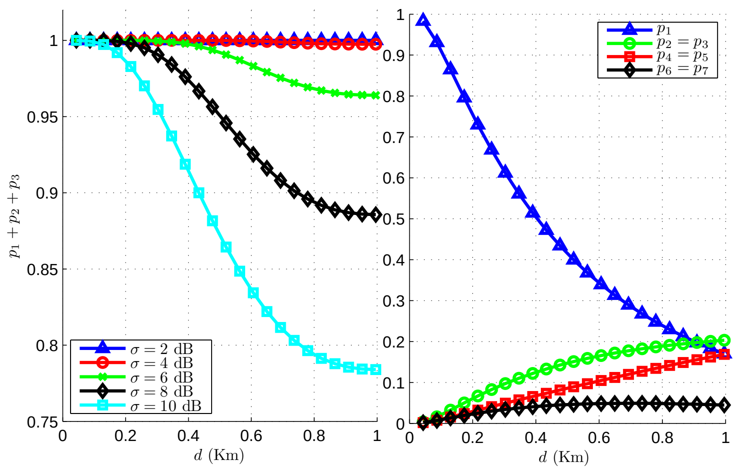

Figure 11: $p_1 + p_2 + p_3$ vs distance d ($\eta = 4$) (left), and p_i for $i = 1...7$ with $\eta = 2.5, \sigma = 10$ dB (right).

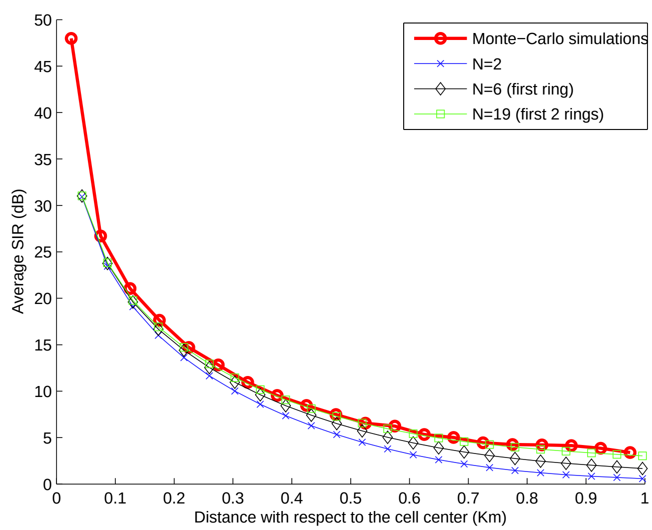

Figure 12: SIR vs. distance to BS, influence of N ($\eta = 2.5$, $\sigma = 10$ dB).

6.6 Best Server vs. Non Best Server

Figure (13) shows the influence of the *best server* policy. Average SIR is compared for the *best server* policy and for the *nearest server* policy as a function of the distance to the nearest BS. Until $d = 500$ m, the nearest BS is also the best one with high probability. At cell border, the *best server* policy provides up to 4.3 dB gain. We can see that the more we get near to the cell border, the more it is important for the MS to be able to choose between more possible serving BSs, because the mean of the received powers becomes comparable. In this case, a high shadowing standard deviation is desirable, because it gives higher probabilities to receive a strong signal from at least one BS.

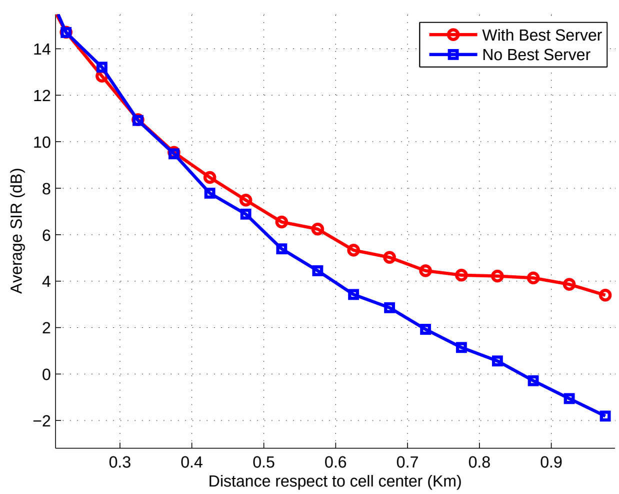

Figure 13: SIR vs. distance to BS, influence of the MS attachment policy (best server vs. non best server) ($\eta = 2.5$, $\sigma = 10$ dB).

7 Conclusion

In this paper, we have proposed an analytical study concerning the average SIR estimation cellular networks in presence of lognormal shadowing, with the realistic *best server* policy. The study makes use of an original approach based on the probablity for and MS to be served by a given BS. Efficient and fast analytical approximations for the estimation of the average SIR and SIR distribution are then introduced. These methods are based on truncated lognormal random variables and several simplificative assumptions. Results show that the analytical approaches gives very good results compared to Monte Carlo simulations, even for high values of the shadowing standard deviation.

References

[1] G.L. Stuber. Principles of mobile communications. *Norwell*, 1996.

[2] J.-M. Kelif and M. Coupechoux. Effect of shadowing on outage probability in fluid cellular radio networks. *WiOpt'08*, Apr. 2008.

[3] J.-M. Kelif and M. Coupechoux. Impact of topology and shadowing on the outage probability of cellular networks. *ICC'09*, Jun. 2009.

[4] A.J. Viterbi, A.M. Viterbi, and E. Zehavi. Other-cell interference in cellular power-controlled CDMA. *IEEE Trans. on Communications*, Feb./Mar./Apr. 1994.

[5] A.J. Viterbi. CDMA - Principles of spread spectrum communication. *Addison-Wesley*, 1995.

[6] C.C. Chan and S.V. Hanly. Calculating the outage probability in CDMA network with spatial poisson traffic. *IEEE Trans. on Vehicular Technology*, Jan. 2001.

[7] T.T. Tjhung and C.C. Chai. Distribution of SIR and BER performance of DS-CDMA systems in lognormally shadowed rician channels. *IEEE Fifth International Symposium on Spread Spectrum Techniques and Applications*, Sep. 1998.

[8] T.T. Tjhung and C.C. Chai. Distribution of SIR and performance of DS-CDMA systems in lognormally shadowed rician channels. *IEEE Trans. on vehicular technology*, Jul. 2000.

[9] S.S. Kolahi. Other cells interferences of power controlled CDMA systems. *IEEE International Conference on Personal Wireless Communicatons*, Dec. 2002.

[10] S.S. Kolahi, A.G. Williamson, and K.W. Sowerby. Other-cell interference in CDMA systems. *Electronics letters*, Sep. 2004.

[11] M. Minelli, M. Coupechoux, and J.-M. Kelif. Average SIR estimation in cellular networks with best server policy. *IEEE Wireless Days Conference*, Oct. 2010.

[12] J.-M Kelif, M. Coupechoux, and P. Godlewski. Spatial outage probability for cellular networks. *GLOBECOM '07*, Nov. 2007.

[13] P.J. Davis and P. Rabinowitz. Methods of numerical integration, 2nd ed., page 51. *Dover Editions*, 2007.

[14] A. Papoulis. *The Fourier Integral and its Applications, Pag. 234-236.* McGraw-Hill, 1962.

[15] S.G. Krantz. Handbook of complex variables. *Boston, MA: Birkhäuser, p. 118*, 1999.

[16] A. Papanicolaou. Taylor approximation and the delta method. `www.stanford.edu/class/cme308/notes/TaylorAppDeltaMethod.pdf`, 2009.

[17] I. Bebu and T. Mathew. Confidence intervals for limited moments and truncated moments in normal and lognormal models. *Statistics and Probability Letters*, 2009.

[18] M. Coupechoux J.-M. Kelif and P. Godlewski. A fluid model for performance analysis in cellular networks. *EURASIP Journal on Wireless Communications and Networking, vol. 2010*, Jul. 2010.

[19] L. F. Fenton. The sum of lognormal probability distributions in scatter transmission systems. *IRE Trans. Commun. Syst., vol. CS-8, pp. 57–67*, 1960.

[20] Recommendation ITU-R M.1225. Guidelines for evaluation of radio transmission technologies for IMT-2000. 1997.